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THE DESCRIPTIVE GEOMETRY
AND THE
PERSPECTIVE
OF
THE STRAIGHT LINE
WITH A
BRIEF INTRODUCTION
TO THAT OF
CURVES
ACCOMPANIED BY MANY EXERCISES.

....BY....
WILLIAM J. MEYERS,
PROFESSOR OF MATHEMATICS,
THE STATE AGRICULTURAL COLLEGE OF COLORADO.

> *"....The imagination....it is material....to fortify and exalt it."*
> *Bacon—The Advancement of Learning.*

FORT COLLINS, COLORADO.
WILLIAM J. MEYERS, PUBLISHER.
1896.

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PREFACE.

The idea which has governed the outlining of the work indicated in the following pages has been that the value of the study of Descriptive Geometry lies, for the ordinary student, chiefly in the discipline which his imagination receives through his familiarizing himself with the fundamental operations of that portion of applied Geometry. My aim has therefore been to strike the happy mean between a treatment of the subject so abstract and difficult as to call forth no clear ideas in the mind of the student, and one so diffuse and easy as to call forth no real mental exertion on his part.

The method herein adopted is one which has been hammered out in the actual work of the class-room. Attention is particularly called to the use made of perspectives of figures embodying the fundamental relations as shown in Plate I. In some respects, the use of perspective representations will be found to be equally efficient with that of models; in others the superiority of models is, of course, undeniable. In almost all cases, however, the student will find it

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advisable to construct for himself a rough model of the first quadrant (better yet of the four quadrants) by nailing two pieces of thin board together so as to form a right diedral (two pieces of heavy card-board will serve fairly well), and then using threads to represent lines, and light cardboard or stiff paper cut to various angles to represent planes. He will find such a rough apparatus as this of considerable aid at the beginning of his study, but should early cast it aside and depend as much as possible on his imagination in tracing the relations of the different parts of the figures.

The chapter on perspective has been added out of deference to custom. In that, as in the preceding chapters, only such development of the subject as was thought to be of value as discipline has been attempted. For any more extensive development of the subject the student is referred to books on the subject prepared especially for architects. Of these, Ware's Modern Perspective, published by James R. Osgood and Co., of Boston, is perhaps as good as any in the English language.

The author will be glad to receive such criticisms and comments on the work herewith offered as may suggest themselves to instructors who may have occasion to use it with their classes.

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NOTATION.

a, b, c, d, e, f,	straight lines or sects.
g,	ground line.
g_1,	first auxiliary ground line, the one ⊥ to H.
g_2,	second auxiliary ground line, the one ⊥ to V.
h,	horizon.
i, j, k, l,	curves.
m,	meridian.
n_q,	the normal through *q*, in perspective.
o,	origin of co-ordinates.
p, q, r, t, u, v, w,	points.
s,	point of sight.
s',	center of picture.
s_1,	foot of station.
x, y, z,	co-ordinates.
A, B, C, D, E, F, I, J, K,	planes.
G,	ground plane, in perspective.
H,	horizon plane in perspective; otherwise the horizontal, or first, base of projection.
L_{pq},	length of the sect joining *p* and *q*.
L_{sa},	length of portion of *a* between its traces on the *first* and *second* bases of projection.
M,	meridian plane in perspective.
N,	plane normal to PP in perspective.
O_3, O_7, etc.,	third octant, seventh octant, etc.
R, U, W, X, Y, Z,	curved surfaces.
P_aH, or hP_a,	projector of *a* on H, or horizontal projector of *a*.
PP,	in perspective, the picture plane.
Q_2, Q_3, etc.	second quadrant, third quadrant, etc.
S,	side, or third, base of projection.

TaV, or vTa,	trace of a on V, or vertical trace of a.
V,	the principal vertical base of projection, also called the second base of projection.
V_q.	in perspective, the secondary plane (or plane parallel to PP) through the point q.
c, c', c''',	projections of c on H, V, and S, respectively.
αa,	the angle that a makes with a'.
βa,	the angle that a makes with a''.
γa,	the angle that a makes with a'''.
$\angle ab$,	the angle that a makes with b.
ΔAB,	the diedral that A makes with B.
$a_{\mathrm{rH}a'}$,	a revolved into H about a' as axis.
$q_{\mathrm{rH}a'}$	q revolved into H about a' as axis.
$a_{\mathrm{r}\parallel\mathrm{H}i}$,	a revolved parallel to H about i as axis.
B_{dH},	B developed into H about TBH as axis.
τ,	the angle between the traces of a plane.
τBHS,	the angle between the traces of B on H and S.

Besides the symbols above noted there are to be used the ordinary geometric symbols for trigon, parallelogram, perpendicular, angle, etc.

CHAPTER I.

DEFINITIONS, NOTATION, ETC.

1.—**DESCRIPTIVE GEOMETRY is that portion of applied geometry which is concerned with the investigation, by means of drawings in a single plane, of the properties of figures not lying in the plane of the drawings.** It is the scientific basis of draughting. It finds application wherever the relations between the different parts of any solid are to be found without first constructing the figure, and in many cases the drawings used in it are far superior to the actual solid figure for purposes of investigation, to say nothing of the great saving of time and labor effected by their use; for instance, if a tin-smith wishes to make two intersecting tubes, he makes use of the principles of this science in laying out the patterns by which to cut his sheets of metal so that when fastened together the desired figure will result. Aside from its direct utility in the industrial arts, descriptive geometry also has great value as a means of mental discipline, its study being perhaps better adapted to the development and discipline of the imagination than that of any other science.

2.—If from any point q, (see fig. 1,) we draw a line through any other point n to meet a given surface R, the point u in which the given line qn meets the surface R is the **projection of n upon R from q as center of projection.** The surface R is the **base of projection;** the line from the center of projection q through the projected point n is the **projecting ray** of n.

3.—**The aggregate of the projections,** upon any surface R, **of the various points of any figure** X from any center of projection q, is called **the projection** of X upon R from q as center; and the **aggregate of the projecting rays** of the various points of X from q as center is called **the projector** of X from q as center.

Q. and E. 1.—What kind of figure is the projection of a straight line upon a plane?

2.—May the projection of a straight line upon a plane ever be merely a point? If so, under what conditions,—if not, why not?

3.—May a straight line ever be so placed as to have no projection on a given plane? If so, under what conditions?

4.—What kind of figure is the projection of a circumference upon a plane?

5.—May a circumference ever have such position that its projection upon a given plane will be a straight line?—a circumference?—an elongated closed curve?—an open curve?—two open curves?—two closed curves?—three open curves?—such that it will have no projection upon a given plane? If any of these questions receives an affirmative answer, state the conditions,—if a negative answer, show why.

6.—What kind of figure is the projection of any plane curve upon a plane when the plane of the curve contains the center of projection? Are there any exceptions? What kind of figure when the plane of the curve does not contain the center of projection?

7.—What kind of figure is the projection of any non-plane curve upon a plane?

4.—The kind of projection just described is what is meant in the higher mathematics by the term "projection" when used without qualification. In this text-book, however, we shall call it **centric projection**, because we wish to use the term "projection" (unqualified) in a more restricted sense.

5.—If the center of projection be taken at an infinite distance from the **projectee** (*i. e.*, **the figure projected,**) the projecting rays within any finite distance from the projectee will be indistinguishable from parallel lines. Under such circumstances we shall call the projection **parallel.**

8.—What kind of figure is the (parallel) projector of any non-plane curve?

6.—If, in the case of parallel projection, the **base** of projection is a **plane perpendicular to the projecting rays**, the projection is called **orthogonic.** Since orthogonic projection is the kind most frequently used, it is to be understood that that is the kind meant by the term "projection" when used without qualification in this text-book.

7.—If two figures, neither of which is contained entirely within the other, have any part common, the common part is called the **trace of either figure upon the**

other; thus the trace of one line upon another is a point, of a line upon a plane is a point; of one plane upon another is a straight line, etc.

9.—What is the trace of a trigonic prism upon a plane to which its lateral edges are not parallel?—of a cylinder on a plane to which its elements are not parallel?—of a cylindric surface upon a plane?—of a spheric surface upon a plane?—of a point upon a plane?

10.—What relation exists between the projection of any figure upon a given base of projection and the trace of its projector upon the same base?

8.—It is evident that the (orthogonic) projectors of any point upon two intersecting planes intersect at the point; and as two straight lines can intersect in but one point, the **projectors of any point upon two intersecting planes determine that point.** These projectors are themselves determined by their traces upon the planes to which they pertain, and these traces are the projections of the point under consideration upon the two planes. **Any point then is completely determined by its projections on two intersecting planes.** Any figure being determined whenever all its points are determined, it is completely determined by the projections of all its points upon two intersecting planes.

9.—It is exclusively with the projections of figures that Descriptive Geometry is concerned, and the **problem of Descriptive Geometry** may be defined to be —**To determine from the projections of any figure the properties of that figure.**

10.—For simplification **the two planes** upon which the projections of any figure are made are usually so taken as to be **perpendicular** to each other, and are always to be understood to be so taken unless the contrary is stated. **One** of the two planes is usually conceived to be **horizontal, the other** thus being **vertical;** for this reason they are called the H and V bases of projection respectively, or, for brevity, the H and V planes, and are denoted by the letters H and V. (See fig. 2.)

11.—The **projection** of any figure **upon** V is called **the vertical projection** of the figure; that **upon** H, the **horizontal projection.**

12.—Two planes perpendicular to each other divide all space into four equal parts; these are called **quadrants.** With respect to H and V, the quadrant above H and in front of V (*i. e.*, on the side of V next to the observer) is called the first quadrant, and is denoted by Q_1; that above H and back of V, the second quadrant, denoted by Q_2; that below H and back of V, the third quadrant, denoted by Q_3; and the remaining one, the fourth quadrant, denoted by Q_4.

13.—The trace of V upon H is called the **ground line,** and is denoted by g. (See fig. 2.)

14.—The **projector** of any figure will be **denoted** by P followed by the symbol of the figure, and the base of projection will be denoted by the letter following the symbol of the figure; thus, P*a*H will denote the projector of *a* upon H, P*a*V the projector of *a* upon V, etc.

15.—In practice it would be very awkward for the draughtsman to be compelled to make part of his drawings on one plane, and part on another; so, in order to avoid this, H and V are conceived to be perpendicular to each other only for the purpose of conceiving the horizontal and vertical projections; and then, before executing the drawings, **the two planes H and V are supposed to be revolved about g as an axis until the two faces of Q_2 come together** as likewise do those of Q_4. **That portion of the drawing-sheet which lies above g thus carries the projections appearing upon the upper part of V and the back part of H**, while that portion lying **below** g carries the projections appearing upon the **lower part of V and the front part of H.** This must always be borne in mind by the student.

11.—In what quadrant is a point which has its vertical projection above g, and its horizontal projection below g?—both projections below g?—both above g?—horizontal above g and vertical below g?—vertical on g and horizontal below g?—above g?—horizontal on g and vertical below g?—above g?

16.—The **confusion** which might be expected to arise from the super-posing of the drawings of two projections upon the same portion of the drawing-sheet is **avoided in practice** by the notation adopted, and by the use of special lines for special purposes. Throughout this text-book we shall use the notation which we shall now explain. **Points** in general (*i. e.*, the points themselves in their actual positions) will be

denoted by the lower-case Italic letters, *m*, *n*, *p*, *q*, *s*, *t*, *u*, *v*, and *w*. **If any point be denoted by** *q*, **its horizontal projection will be denoted by** q', and its **vertical projection** by q''. **Lines** will be **denoted** by the lower-case Italic letters, *a*, *b*, *c*, *e*, and *f*; **curves** by *i*, *j*, *k*, and *l*; the horizontal projections by a', b', c', etc., respectively, and the vertical projections by a'', b'', c'', etc., respectively. The **trace** of the line *a* upon H will be **denoted** by T*a*H (read "trace of *a* upon H"), or by hT*a* (read "the horizontal trace of *a*"), as may be the more convenient; that upon V by T*a*V, or by vT*a*, etc. **Planes** will be **denoted** by the capital Italic letters, *A*, *B*, *C*, *D*, *E*, *F*, *I*, *J*, and *K*. The trace of *A* upon H will be denoted by T*A*H or by hT*A* as may be the more convenient; that upon V by T*A*V or by vT*A*; that upon g by T*A*g. The **complete trace** of the plane *A* upon the two planes of projection will be **denoted** by T*A*HV or by hvT*A*. **Curved surfaces** will be **denoted** by the capital Italic letters, *R*, *U*, and *W*. The trace of the surface *U* upon H will be denoted by T*U*H or by hT*U*, etc. With such exceptions as may hereafter be noted, **plane angles** will be **denoted** by the lower-case Greek letters of the first part of the alphabet, **diedrals** by capital Greek letters of the first part, and **polyedrals** by capital Greek letters of the latter part of the alphabet.

17.—With regard to **figures lying in** Q_1, the **traces** of **given planes** and the **projections** of **given lines** are **drawn full,**—thus, ————————; those of **required planes and lines** are **composed of long dashes,**—thus, —— —— ——. **Given points** are **denoted by small crosses,** and **required points have very small circles drawn about them.** Planes and lines neither given nor required, but used to aid in the solution of any problem are called **auxiliary;** auxiliary planes have their traces composed of dashes of medium length with two dots or very short dashes in each interval,—thus,——··——··——, while auxiliary lines have their projections composed of dashes of medium length alternating with single dots or very short dashes,—thus, ——·——·——. With regard to **figures in** Q_2, Q_3, and Q_4, traces of given planes and projections of given lines are drawn with dashes of medium length, and these are usually made fainter than the corresponding full lines for figures in Q_1. Other lines are drawn as for figures in Q_1, but usually much fainter. The **projections of the same point** are usually **connected** by lines composed of dots or of very short dashes.

18.—**Co-ordinates.** In locating the position of a point, we shall use the letter y to denote its distance from H, taking y to be algebraically positive when the point is above H, negative when it is below H; z will be taken to denote its distance from V, positive when the point is in front, negative when it is back of V. It will be noticed that the position of a point is not completely determined by its distance from H and V, but merely that of some line on which the point must be. In order to define the point completely, we must know its distance from a third plane not parallel to the intersection of the other two. Such a plane is, for simplicity, taken perpendicular to g and passing through some definite point of it. This plane is called, in Descriptive Geometry, a side plane and we shall denote it by S. The trace of S upon V is called the first auxiliary ground-line and is denoted by g_1; that of S upon H, the second auxiliary ground-line, denoted by g_2. The distance of a point from S is denoted by x, and is called positive when the point is to the right-hand of S, and negative when to the left-hand. These three distances of the point, denoted by x, y, and z, taken with regard to their algebraic signs, are called the co-ordinates of the point. The planes from which they are measured, S, H, and V, are called the co-ordinate planes. Their intersections are called the co-ordinate axes; g, the x axis; g_1, the y axis; and g_2, the z axis. The point common to the axes is called the origin of co-ordinates and is denoted by o. The side plane S is also used in Descriptive Geometry as a base of projection, but this use of it will be discussed later. In denoting the position of a point, the values of the co-ordinates are written in the order x, y, z, separated by commas, and enclosed in a parenthesis; thus, the point $(2, -3, 5)$ is the point for which $x=2, y=-3$, and $z=5$. In the exercises called for in connection with co-ordinates, the student may locate the point o on g, but need not attempt to show the plane S, the auxiliary ground lines, or the side projections of the points.

19.—**Octants.** The three planes, S, H, and V, divide all space into eight equal parts called octants. These are numbered similarly to the quadrants. The first four are those to the right-hand of S, and each one to the left-hand comes fourth in order after the one immediately to the right of it. They are denoted O_1, O_2, etc.

12.—Give the octant to which each of the following-named points belongs, and say how far each point is from each base of projection;— $(1,-4,6)$, $(3,7,-9)$, $(-2,5,8)$, $(-3,-5,7)$, $(4,2,3)$, $(5,-6,-1)$, $(-2,4,-7)$, $(-8,-3,-5)$, $(0,6,1)$, $(0,-1,2)$, $(0,1,-3)$, $(0,-2,5)$, $(3,0,6)$, $(3,5,0)$, $(-2,4,0)$, $(4,-3,0)$, $(-2,0,-3)$, $(-3,-6,0)$, $(-6,0,-4)$, $(5,0,-8)$, $(-4.0,12)$, $(3,0,0)$, $(0,-7,0)$, $(0,0,4)$, $(0,6,0)$, $(-5,0,0)$, $(0,0,-1)$, $(0,0,0)$.

20.—When S is used as a base of projection in Descriptive Geometry, before drawings are executed upon it, it is revolved about g_1 so that the two vertical faces of O_5 and of O_8 come together, as also do those of O_2 and those of O_3.

CHAPTER II.

FUNDAMENTAL PROPOSITIONS.

Certain fundamental propositions to which we shall have occasion to refer later are here stated and briefly elucidated where necessary. The student should note them carefully.

21.—Proposition I. **If q be any point, PqH is parallel to V, and PqV is parallel to H.** (See fig. 2.)

22.—Cor. I. **The distance of q' from g equals the distance of q from V, and the distance of q'' from g equals the distance of q from H.** (See fig. 2.)

23.—Cor. II. **If q lies in V, q' lies in g, and conversely.** So also **if q lies in H, q'' lies in g, and conversely.**

24.—Cor. III. **If a line is parallel to either base of projection, its projection upon the other is parallel to g, and conversely.** (See fig. 6.)

25.—Cor. IV. **If any plane figure is parallel to either base of projection, its projection upon the other is a straight line parallel to g, and conversely.** (See fig. 5.)

13.—Say in which quadrant q is and at what distances from H and V if

q' is 3 inches above g and q'' is 2 inches above g
q' is 2 inches above g and q'' is 2½ inches above g
q' is 2 inches below g and q'' is 5 inches above g
q' is 7 inches below g and q'' is 4 inches below g
q' is 1½ inches above g and q'' is 3½ inches below g
q' is on g and q'' is 2 inches below g
q' is on g and q'' is 4 inches above g
q' is 6 inches above g and q'' is on g
q' is 8 inches below g and q'' is on g.

14.—Give the location on the drawing sheet of each projection of each point in ex. 13, taking first point 1¼ inches to left of S, next 1 inch to left, next ¼ inch to left, etc.

26.—Proposition II. **If q be any point, the line joining q' and q'' is perpendicular to g. Conversely, if two given points, one in H and one in V, are on a line perpendicular to g, there is some point of which the given points are respectively the horizontal and vertical projections.**

First,—Pass a plane through PqH and PqV. It is ⊥ to H and to V, ∴ to g. Therefore its traces on H and V are both ⊥ to g. But when H and V are revolved into their customary positions for drawing, these two traces form a single line ⊥ to g and passing through q' and q''.

Second,—If u be a point of H and w a point of V, and the line uw be ⊥ to g, intersecting it (when H and V are in position for drawing) at t, the two lines tu and tw, when H and V are restored to their ideal positions, determine a plane ⊥ to g, ∴ ⊥ to H and to V. Lines erected at u and w ⊥ to H and to V respectively will lie in the plane utw, and being ⊥ to intersecting planes will be non ∥. They will ∴ meet, and their point of intersection is the point of which they are the projectors, and of which u and w are respectively the horizontal and vertical projections. (See fig. 2.)

27.—Proposition III. **If the point q is on the line**

a, q' **is on** a', **and** q'' **is on** a''. **Conversely, for every point on any projection of a line, there is a corresponding point on the line.** (See fig. 2.)

28.—Proposition IV. **The projection of a straight line is always determined by the projections of any two of its points:** for when a straight line is ⊥ to the base of projection, its projection is a point, and in all other cases it is a straight line.

29.—Proposition V. **If both projections of any line are straight, the line is straight:** for in this case the two projectors are both plane, and the intersection of the projectors of any line determines the line. The student may determine whether or not there is any case in which this proposition fails.

30.—Proposition VI. **If one of the projections of a line is perpendicular to** g, **the line lies in a plane perpendicular to** g, **and its projections are therefore both perpendicular to** g **and meet it at the same point.** (See fig. 3.)

31.—Cor. **If at two different points on** g **perpendiculars be erected, one in** H, **the other in** V, **no line can be found of which they are the projections.**

32.—Proposition VII. **Projections of parallel lines upon the same plane or upon parallel planes are parallel.** (See fig. 4.)

33.—Proposition VIII. **If a plane is perpendicular to either base of projection, its trace upon the other is perpendicular to** g, **and conversely.** (See fig. 4.)

34.—Proposition IX. **The two traces of any plane upon H and V (called respectively the horizontal and vertical traces of the plane) meet g at the trace of the plane upon g.** (See fig. 4.)

35.—Cor. I. **If a plane is parallel to g, its traces are parallel to g, and conversely.**

36.—Cor. II. **If the line *a* lies in the plane *A*, the trace of the plane *A* upon any other plane *B* passes through the trace of the line *a* upon the plane *B*.** (See fig. 6.)

37.—Proposition X. **Traces of parallel planes upon the same plane are parallel.** (See fig. 6.)

38.—Cor. I. **If the horizontal traces of two planes are parallel and their vertical traces are parallel, the planes are parallel, provided the traces are not parallel to g.** (See fig. 6.)

39.—Cor. II. **If the horizontal projections of two lines are parallel and their vertical projections are parallel, the lines are parallel.** (See fig. 4.) The student may determine whether or not there is any case in which this corollary fails.

40.—Proposition XI. **If a plane and a line are perpendicular, the trace of the plane and the projection of the line upon any second plane are perpendicular.**

Call the plane *A*, and the line call *b*, and the plane upon which the projection takes place call H. Then P*b*H is $\perp$ to *A* and to H, ∴ to their intersection T*A*H. ∵ T*A*H is ⊥ to P*b*H, it is ⊥ to the trace of P*b*H upon H; but this trace is b', ∴ b' and T*A*H are ⊥ when b is ⊥ to A.

41.—Cor. **If one of the two sides of a right angle is parallel to a plane, the projection of that right angle upon that plane is itself a right angle.**

Call the two lines a and b, and suppose them to meet at the point q and to include a right ∠. Then if the line a is ∥ to H, the lines a' and b' include a right ∠. For, through q pass a plane ⊥ to a; this plane will contain the line b and will be ⊥ to H, ∵ it is ⊥ to a which is ∥ to H. It will ∴ be P*b*H, and its trace upon H will ∴ be b', and this trace, by the preceding proposition will be ⊥ to a'.

42.—Proposition XII. **Any plane figure is equal to its projection upon any plane to which the figure is parallel.** (See fig. 5.)

43.—Proposition XIII. **The projections (upon the same plane) of any two homothetic figures are homothetic, and their center of homothesy is the projection of the center of homothesy of the two given figures.** (See fig. 8, left-hand portion.)

44.—Proposition XIV. **If a straight line a is tangent to a curve k at the point q, then a' is tangent to k' at the point q'.** (See fig. 8, right-hand portion.)

45.—Cor. **If a curve k is tangent to another curve l at the point q, then k' is tangent to l' at q'.**

CHAPTER III.

FUNDAMENTAL OPERATIONS.

The various fundamental operations of Descriptive Geometry will be best understood by carefully considering their application in certain simple and elementary problems. These will now be stated and discussed.

46.—Problem I. **Of a line a, given a' and a'', and of q, a point on a, given q', to find q''.**

Analysis q'' is on a'', and the line joining q' and q'', is $\perp$ to g. $\therefore q''$ is where a $\perp$ from q' upon g crosses a''. Likewise when q'' is given to find q'. (See fig 2.)

47.—**Digression upon the use of S.** It is evident that when a in problem I is $\perp$ to g, the method there developed is inapplicable, $\because a'$ and a'' are both $\perp$ to g. It becomes necessary, then, to make use of some other base of projection whose ground line is such that a is not $\perp$ to it. For this purpose we usually employ S as the new plane, and use it in conjunction with both H and V. Projections upon S are denoted by three accents; thus, the projection of a upon S is denoted by a'''. Evidently q''' is as far from g_1 as q is from V, and $\therefore$ as q' is from g, while q''' is as far from g_2 as q is from H, and $\therefore$ as q'' is from g. Given q' and q''

then, we may easily construct q''' when g_1 and g_2 are known. The student may readily trace out by means of the perspective figure in the left-hand part of fig. 3, the use of the side plane in the solution of problem I.

48.—Problem II. **Given a' and a'' of a line a, to find T*a*H and T*a*V.**

Analysis. T*a*H, being a point of *a*, is vertically projected in a'', and being a point of H is vertically projected in g. ∴ it is vertically projected where a'' meets g. T*a*H, being a point in H, is its own horizontal projection, and must ∴ be on a'. Since (T*a*H)″ is known, T*a*H is readily found by means of the method developed under problem I. Likewise for T*av*, and for T*a*S. In the case where *a* is ⊥ to g, S would be used. (See figs. 2 and 3.)

49.—Problem III. **Of a line a and a point q not on a, given a', a'', q', and q''; to draw through q a line b which shall be parallel to a.**

Analysis. b' must pass through q' and be ∥ to a'; likewise for b'', q'', and a''.

50.—Problem IV. **Of a sect a, given a' and a'', to find the length of a, and the angle *aa* which a makes with a'.**

Analysis. If a' is ∥ to g, *a* is ∥ to V and a'' shows the true length of *a*, and the ∠ which a'' makes with g = *aa*. If a' is not ∥ to g, let *m* and *n* (see fig. 7) denote the ends of *a*. Suppose through one end of *a*, *m* say, a line to be drawn ∥ to a' and meeting P*n*H in *q*. Then the ∠*nmq* = *aa*, and the hypotenuse *mn* of the rt. △ thus formed is *a*. Now suppose this rt. △*mqn* to be revolved about P*n*H as an axis until ∥ to V. *m* will trace a circular $\frown$, centered at *q*, ∥ to H, and terminating at a point (call it *t*) equidistant with *q* from V. $(\frown mt)'$ will ∴ be a circular $\frown$, centered at q', having a radius = to a', and terminating at t', $q't'$ being ∥ to g; and $(\frown mt)''$ will be a straight line ∥ to g. t'' may thus be readily found, and $t''n''$ gives the true length of *a*

Analysis. Through any point q of b draw an auxiliary line f ∥ to c. The plane of b and f is A. Notice that if either b or c is ∥ to H, it is ∥ to T_{A}H.

To **pass a plane through a point ∥ to two given lines** is an easy modification of Problem IX. So, also, is it **to pass through a given point a plane ∥ to a given plane** for the traces of the given plane are lines to which the required plane must be ∥. Another simple modification is the problem **to pass through a given point a plane ⊥ to a given line,** for the traces of the required plane must be ⊥ to the corresponding projections of the given line.

56.—Problem X. **To find the trace of one given plane A upon another given plane B.**

Analysis. ∵ T_{AB} is a straight line, it will be sufficient to find two points of it. One such point is evidently where T_{A}H crosses T_{B}H, and another where T_{A}V crosses T_{B}V. (See fig. 11.)

In the case where the traces of the given planes upon one of the bases of projection do not meet within the limits of the drawing, it becomes necessary to resort to other methods than the one above indicated. In this case we customarily use auxiliary planes ∥ to the bases of projection. These of course cut out of the given planes lines ∥ to their traces upon the bases to which the auxiliary planes are ∥, and the intersections of these auxiliary lines give points upon the line sought. When no other method suffices, we resort to a "reduction of scale"; *i. e.*, to the construction of a miniature figure similar to the given one, and after performing the desired operations on this, enlarging the results in the proper ratio. Figures 12 and 13 will illustrate these methods. Figures 14–16 illustrate two useful methods of drawing through a given point a line to concur with two given lines whose point of concurrence lies outside the limits of the drawing, a problem which sometimes arises in such problems as the one just discussed.

57.—Problem XI. **To find the trace of a given line a upon a given plane B.**

Analysis. If through a any auxiliary plane C be passed, T_{CB}

will intersect *a* and will lie wholly in *B*. Therefore the point where *a* crosses T_{CB} is the T_{aB}. Customarily we take for *C* one of the projectors of *a*. (See fig. 7.)

58.—Problem XII. **To find the angle between a given line *a* and a given plane *B*.**

Analysis. Through any point of *a* pass an auxiliary line *c* ⊥ to *B*. Then $\angle ac = 90° - \angle Ba$. (See fig. 17.)

59.—Problem XIII. **To find the projection of a given sect *a* upon a given plane *B*.**

Analysis. The projection of *a* upon *B* is the straight line joining the projections of its extremities on *B*. These may be found by drawing from its extremities ⊥s to *B*, and finding their traces upon *B*.

60.—Problem XIV. **To find the plane angle of the diedral between two planes.**

Analysis. The ∠ sought is the ∠ between the traces of the given planes upon any auxiliary plane ⊥ to their intersection. Or, two lines from any point within the diedral ⊥ to its two faces contain an ∠ supplementary to the ∠ sought. (See figs. 18 and 19.)

61.—Problem XV. **To find the distance from a given point *q* to a given plane *B*.**

Analysis. The distance of *q* from *B* = the distance of *q* from T_{CB}, *C* being any auxiliary plane containing *q* and ⊥ to *B*. In the application of the method here indicated it is customary to take *C* ⊥ to T_{BH} or to T_{BV}. (See fig. 20.)

The distance sought also = the distance from *q* to its projection on *B*. (See fig. 21.)

The problem **to find the distance between ∥ planes** is a simple modification of Problem XV.

62.—Problem XVI. **To draw the shortest line joining two given windschief lines, i. e., two non-planar lines.**

Analysis. Call the two windschief lines a and b, and suppose that the shortest line joining them meets a in r and b in t. Then rt must be ⊥ both to a and to b, for if it were oblique to one of them, b say, then a ⊥ from r upon b would be shorter than rt. Suppose a plane H passed ∥ to a and b and another plane V be taken ⊥ to H; then a' and b' cannot be ∥, ∵ a'' and b'' are ∥, both being ∥ to g, ∵ a and b are both ∥ to H and are not ∥ to each other. At the point where a' meets b', m say, erect a ⊥ to H. Since this ⊥ lies in both PaH and PbH, it will cross both a and b. Since it is ⊥ to a' which is ∥ to a, and to b', which is ∥ to b, the ⊥ to H erected at m is ⊥ both to a and to b. Therefore that portion of it comprehended between a and b is the line sought.* Therefore, to solve Problem XVI, pass through one of the given lines an auxiliary plane parallel to the other, project that other upon it, and where its projection upon this plane crosses the first line erect a ⊥ to the plane and prolong it until it meets the second line. This will be the line sought. (See fig. 22.)

63.—**Development.** To develop means primarily to unwrap, or to unfold. The word is used in Decriptive Geometry to denote the spreading out flat of any surface capable of being so spread without distortion. It is evident that all surfaces whose parts are plane are capable of being developed after separation into their component plane parts, while those in which no straight line can be drawn are as evidently incapable of it; thus the surface of a prism and that of a pyramid are both developable while that of a sphere is not. Of those curved surfaces generated or generable by the motion of straight lines, those are developable in which consecutive positions of the generator are either parallel or concurrent; the others are not; thus those of the cone and of the cylinder are developable, while the hyperboloid of revolution (that generated by revolving one windschief line about another as axis) is not.

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*That there is but one solution to this problem appears from the fact that if rt be prolonged it must be perpendicular to two lines drawn through its trace upon the plane H of the discussion ⊥ to a and b respectively, and ∴ perpendicular to H, ∵ lines drawn through any point of H ∥ to a and b lie wholly in H. rt being perpendicular to H lies in both PaH and PbH, which being non-coincident can have but one common line.

64.— From the use of the term "develop" to denote the spreading or laying out flat of any surface, and the fact that this development is usually effected in H or V, the revolution of a plane into H or V about its horizontal or vertical trace as axis, is commonly called the development of the plane into H or V, and the position taken by any figure of the plane when the plane is so developed, we call the developed position of the figure.

65.—**Intersection of Surfaces.** The intersection of two planes has already been discussed in problem X. When it is desired to find the trace of a plane upon a curved surface, or of one curved surface upon another, we customarily proceed by passing auxiliary surfaces (usually planes) so as to cut both given surfaces. The points common to the traces of the auxiliary surface upon the given surfaces are points of the line sought. By the determination of a sufficient number of such points the line may be determined as accurately as may be desired. The problem of finding the trace of one surface upon another is thus reduced to that of finding the trace of a plane upon any surface. The mode of doing this depends entirely upon the nature of the surface. In the case of surfaces generated by straight lines the auxiliary planes would usually best be taken so as to cut the surfaces along their rectilinear elements if possible; in the case of surfaces of revolution they would best be meridian planes, *i. e.*, planes containing the axis of revolution; planes perpendicular to the axis are also frequently useful as they give circular traces which may readily be drawn, etc. A few instances of the general problem will be given among the exercises in the next chapter.

CHAPTER IV.

EXERCISES FOR DRAUGHTING PRACTICE.

In the exercises below given, **accuracy is the first essential, and neatness is the next.** Without neatness there can be no great degree of accuracy. Great care should be taken to have each line show as nearly as possible by its character just what it is intended to represent, and literal symbols should be employed on the drawing only to denote what cannot be shown by the lines themselves.

The student is particularly cautioned to bear continually in mind **that all work in Descriptive Geometry is performed wholly by means of projections.** Whenever any figure, then, is said to be given, it is meant that the projections, traces, etc., of that figure are given; and whenever any figure is said to be required, it is meant that the projections, traces, etc., of that figure are required.

15.—Draw the three projections of each point given in ex. 12, using one-fourth of an inch as the unit of length, and denoting the different projections by the proper accents.

16.—(i) Given q' and q'', find q'''

(ii) Given q'' and q''', find q'

(iii) Given q''' and q', find q''

17.—Of a line a oblique to g, given a' and a'' and of a point q on a, given q' to find q''—

(i) When a crosses Q_1 and q is in Q_1;

(ii) When a crosses Q_2 and q is in Q_3;

(iii) When a crosses Q_3 and q is in Q_2;

(iv) When a crosses Q_4 and q is in Q_4.

(v, vi, vii and viii,) same as above except a joins n and w (n', n'', w' and w'' being given,) which are in a plane $\perp$ to g and in (v) are in Q_1, in (vi) are in Q_2, etc. For these last four problems use S.

18.—Given n', n'', p' and p'' to find Lnp.

(i) n in Q_1, p in Q_1;

(ii) n in Q_2, p in Q_2;

(iii) n in Q_3, p in Q_4;

(iv) n in Q_4, p in Q_2.

19.—Given n', n'', p' and Lnp, to find p''; how many solutions?

20
21
22
23

GIVEN FIND
$a', a'', \ldots a'''$, TaH, TaV, TaS, $\alpha a, \beta a, \gamma a$, L_3a, L_1a, and L_2a.

Remarks. In examples 20, 21, 22, and 23, a is oblique to S and crosses Q_1, Q_2, Q_3, and Q_4, respectively.

24
25
26
27

GIVEN FIND
$a', a'', \ldots a'''$, TaH, TaV, TaS, $\alpha a, \beta a, \gamma a$, L_3a, L_1a, and L_2a.

Remarks. In examples 24, 25, 26, and 27, a is parallel to S and crosses Q_1, Q_2, Q_3, and Q_4, respectively.

28.—Given a'' and a''', to find a'.

29.—Given a', TaH, αa, find a''. Get all possible solutions.

30.—Given a', TaH, γa, find a''. Get all possible solutions.

In examples 31 to 48 inclusive, get all possible solutions.

GIVEN. FIND. REMARKS.

31.—Given a'', TaV, L_2a, find a'.

32.—a''', TaV, γa....a', a''.

33.—a', TaS, βa......a''.

34.—a''', TaS, L_2a ..a', a''.

35.—a', TaV, L_3a ....a''.

36.—a'', TaS, L_2a....a''.

37.—a', γa, L_2a......a''.

38.—a'', αa, L_3a......a'.

39.—TaH, TaS.......a', a''.

40.—TaV, αa, βa.....a', a''.

41.—TaH, αa, L_3a....a', a''.

42.—TaH, βa, L_2a....a', a''.

43.—αa, βa, L_3a, q' ..a'; a''. q is on a.

44.—αa, βa, L_1a, q''...a', a''. q is on a.

45.—αa, βa, L_3a, L_2a..a', a''.

46.—a', q'', αa.......a''. q is on a.

47.—a', q'', γa ......a''. q is on a.

48.—a', q'', L_1a......a''. q is on a.

49.—TAH, TAV......TAS, ΔAH, etc., τAHV, etc. [Solve also when A is $\perp$ to S.

50.—TAH, ΔAH......TAV. Take Δ acute in one instance, [obtuse in another.

51.—TAV, ΔAS ......TAH. Take Δ acute in one instance, [obtuse in another.

52.—TAH, τAHV .. .TAV. Take τ acute in one instance, [obtuse in another.

53.—TAH, TAV, a'...a''. a is in A. Take τ acute in one [instance, obtuse in another.

54.—TAV, a', a''.....TAH. a is in A. Take a crossing Q_2.

55.—TAV, TAS, q'...q''. q is in A. Take q in Q_3.

56.—q', q'', αa, βa....a', a''. q is in a.

57.—TAH, TAV, q'...q_{r}H$\mathrm{T}_{A\mathrm{H}}$. q is in A.

58.—TAH, q', q''... .TAV, q_{r}H$\mathrm{T}_{A\mathrm{H}}$. q is in A.

GIVEN.	FIND.	REMARKS.
59.—T*A*V, q'', Δ*A*V	T*A*H, $q_r\mathrm{VT}_{AV}$.	q is in *A*.
60 —T*A*H, q', $q_r\mathrm{HT}_{AH}$	T*A*V, q''.	q is in *A*.
61.—T*A*V, q', $q_r\mathrm{VT}_{AV}$	T*A*H, q''.	q is in *A*.
62.—T*A*V, q', τ*A*HV	$q_r\mathrm{VT}_{AV}$.	q is in *A*.
63.—q', q'', $q_r\mathrm{VT}_{AV}$	T*A*H, T*A*V.	q is in *A*.
64.—q'', $q_r\mathrm{HT}_{AH}$, Δ*A*H	T*A*H, T*A*V.	q is in *A*.
65.—q'', q_rV, Δ*A*H	T*A*H, T*A*V.	q is in *A*.
66.—q', $q_r\mathrm{HT}_{AH}$, τ*A*HV	T*A*H, T*A*V.	q is in *A*.
67.—q', $q_r\mathrm{VT}_{AV}$, Δ*A*V	T*A*H, T*A*V.	q is in *A*.
68.—T*A*H, T*A*V, a'	$a_r\mathrm{VT}_{AV}$.	a is in *A*.
69.—a', a'', b', b''	T*C*H, T*C*V.	a and b intersect, and [lie in *C*.
70.—a', a'', b', b''	T*C*H, T*C*V.	a and b are ∥ and lie [in *C*.
71. —a', a'', q', q''	T*C*H, T*C*V.	a and q are in *C*, q is [not on a.
72.—a', a'', b', b''	T*C*H, T*C*V.	a and b are in *C*, but T*b*HV are without the limits of the drawing.
73.—q', q'', t', t'', w', w''	T*C*H, T*C*V.	q, t, and w are in *C*.
74.—q', q'', t', t'', w', w''.	Find the true figure qtw without finding traces of plane of qtw.	
75.—T*A*H, T*A*V, b', b''	∠*bA*.	b pierces *A*.
76.—T*A*H, T*A*V, T*B*H, T*B*V	T*BA*.	
77.—T*A*H, T*A*V, b', b''	T*bA*.	b pierces *A*.
78.—T*A*H, T*A*V, q', q''	distance from q to *A*.	q is not in *A*.
79.—T*A*H, T*A*V, T*B*H, T*B*V	Δ*AB*.	
80.—T*A*H, T*A*V, T*AB*, Δ*AB*	T*B*H, T*B*V.	
81.—T*A*H, T*A*V, T*B*H, T*B*V, q', ∠*aB*	a', a''.	[a is in *A*, q is on a.
82.—T*A*H, T*A*V, T*B*H, T*B*V, q', q'', ∠*aB*	a', a''.	q in neither *A* nor *B*; a goes through q and is ∥ to *A*.
83.—T*A*H, T*A*v, b', b'', q', q''	T*C*H, T*C*v.	q in neither *A* nor *B*; *C* contains q, is ⊥ to *A*, and ∥ to b.
84.—a', a'', T*B*H, T*B*v, Δ*CB*	T*C*H, T*C*v.	*C* contains a.

Triedrals. α, β, and γ are face angles; A, B, and Γ are the diedrals opposite α, β, and γ respectively; and a, b, and c are the edges of the diedrals A, B, and Γ respectively. w is the vertex. In each exercise find the other parts, and where sums or differences are given, find the component parts also. The student should notice the analogy between the relations holding among the parts of triedrals, and those among the corresponding parts of trigons.

GIVEN.
85.—α, β, γ
86.—α, β, Γ
87.—α, β, A
88.—α, B, Γ

GIVEN.
89.—α, B, $\beta+\gamma$
90.—α, B, $\beta-\gamma$
91.—A, B, Γ
92.—α, A, B.

GIVEN.
93.—α, β, A+B
94.—α, β, A−B
95.—α, B, A−Γ

96.—Given an oblique prism with a scalene trigonic base, edges oblique to H and to V. Find the projections of the trace of the prism upon a plane oblique to H, to V, and to the edges of the prism, also the true figure of the trace.

97.—Given an irregular tetragonic pyramid. Find the projections, also the true figure, of the trace of the pyramid on a secant plane oblique to H and to V.

98.—Develop the lateral surface of the truncated prism between H and the secant plane in problem 96 above; also that of the truncated pyramid between H and the secant plane in problem 97.

99.—An oblique trigonic prism cuts into but does not pierce an oblique tetragonic prism. Find the trace of one on the other. Develop the surface of each prism, so as to find patterns by which the surfaces might be cut out of sheet-metal or card-board. Cut them out of card-board and see whether or not they can be made to fit properly.

100.—Having given the three lateral edges of a trigonic pyramid, no two of them being equal, construct the pyramid so that the face angles shall have given sizes.

101.—Draw a trigonic pyramid whose face angles at the vertex shall have given sizes, no two being equal, and whose lateral faces shall make equal diedrals with its base.

102.—Draw a trigonic pyramid whose lateral edges shall have given lengths, no two being equal, and whose lateral faces shall make equal diedrals with its base. How many solutions?

103.—Given a tetragonic pyramid having a trapezium for base, to cut it by such a plane that the section shall be a parallelogram.

104—Having given three of the lateral edges, no two being equal, of a tetragonic pyramid, draw the pyramid so that the face angles at its vertex shall be equal and the diedrals between consecutive lateral faces shall be equal.

105.—Given a pentagonic pyramid with its base in the rear part of H, and its vertex in Q_1. Find the shadow cast on H, also that cast on V, by rays of light emanating from a given point in Q_1, nearer to V than is the vertex of the pyramid. and above and to the left of it.

106.—The same as in problem 105, except that the rays of light are parallel, and their direction is given.

107.—A cylinder whose base is an irregular curve in H, and whose elements are oblique to H, to V, and to S, is cut by a plane perpendicular to its elements. Find the projections, also the true figure, of the trace of the plane on the cylinder. Develop the curved surface of the truncated cylinder between H and the secant plane.

108.—An oblique circular cone whose base is in H and whose vertex is nearer V than is the center of its base, is cut by a plane oblique to H and to V. Find the projections, also the true figure of the trace of the plane on the cone. Develop the curved surface of the truncated cone between H and the secant plane.

109.—Find the shadow of a cone upon a cylinder. The base of the cone is irregular and lies in H; the cylinder has a circular right section and its axis is parallel to H and oblique to V; the rays of light are parallel, and the shadow of the apex of the cone falls on the curved surface of the cylinder.

110.—A cone of irregular base pierces a cylinder of irregular base. Find the projections of the trace of the cone on the cylinder.

111.—A given oblique plane cuts a given sphere. Find the projections of the trace of the plane on the sphere, also the true figure of the trace. Find also the shadow of the sphere on the plane, the rays of light emanating from a point on the same side of the plane as is the center of the sphere.

112.—Find the projections of the trace of a sphere upon a right circular cone. The apex of the cone is just without the sphere, and the trace of the conic surface upon the spheric surface consists of two closed non-circular curves. Develop the surface of the cone and find the developed trace.

113.—Two cylinders of circular right section are respectively two inches and three and one-half inches in diameter, and their axes are inclined to each other at an angle of 30°, and where nearest are one-half inch apart. Find the projections of the trace of one upon the other, and develop the surface of each, showing the developed trace on each.

114.—Find the projections and the true figure of the trace of an oblique plane upon an hyperboloid of revolution. The hyperboloid has its axis vertical and is obtained by revolving about the axis a straight line distant from it one inch and making with it an angle of 45°.

115.—Given q', q'', $\Delta A\mathrm{H}$ and $\Delta A\mathrm{V}$, to draw A through q.

ABBREVIATIONS.

The student will find the following list of abbreviations useful in taking notes in geometry. They are used in the notes which follow.

adj	adjacent	ctr	center	g^n	given
alt	altitude	cv	curve	h	horizontal
apl	apply	cyl	cylinder	hpt	hypotenuse
argt	argument	cylc	cylindric	ht	hight
aux	auxiliary	d	draw	hyp	hypothesis
bet	between	Δ	diedral	inc	increment
bs	bisect	dg	diagonal	incd	include
bsr	bisector	dm	diameter	incl	incline
cdr	consider	dir	direct	int	interior
cf	compare *or*	dnt	denote	isn	intersection
	" with	dp	drop	isos	isosceles
cmpt	complement	ds	desire	ist	intersect
cmpv	comparative	dscs	discuss	k	know
cmpn	comparison	dst	distant *or*	l	long *or*
cncd	coincide		distance		length
cncr	concur	dt	determine	lc	locate *or*
cncv	conceive	dvp	develop		locus
cnst	consist	el	element	lim	limit
cntg	contiguous	eql	equilateral	lt	lateral
cntn	contain	es	essential	max	maximum
cnv	convenient	exp	explement	min	minimum
com	common	f	find	mntn	mention
crs	cross	fig	figure	mthd	method
crsp	correspond	gm	geometry *or*	mtl	mutual
cstr	*construct*		geometric	n	notice

nb	notice well
neg	negative
ny	necessary
o	origin
obl	oblique
ol	original
opp	opposite
p	page
pb	problem
pcd	proceed
pgn	polygon
pj	project
pl	plane
pm	prism
pos	positive
prin	principle
ps	pass
psn	position
pt	point
ptn	portion
pyr	pyramid
r	revolve
rad	radius *or* radiate
rd	reduce
rl	relation *or* relative
rm	remain
rq	require
rsp	respective
rt	right
rvrs	reverse
ry	readily
s	side
sec	secant
seg	segment
sgst	suggest
sh	show
sim	similar
sl	slant
slh	slant hight
sln	solution
slv	solve
sm	small
sn	section
sp	suppose
sph	sphere
spr	superior
sstn	sustain
sup	supplement
surf	surface
t	take
tan	tangent
thr	through
tr	trace
trnc	truncated
u	usual
v	vertical
vx	vertex
wh	which, whose *or* where

Suffixes to these words will be indicated by the use of the last letter of the suffix as a superior affixed to the abbreviation; thus, d^{n} indicates drawn,—pl^{s}, planes, etc.

NOTES ON PROBLEMS.

20–27.—a' and a'' being g^{n}, the s pj^{ns} of the ends of a may ry be d^{n}, ∴ a'''. Pb II (art. 48, p 17) dsc^{s} the mode of 1^{g} TaH and TaV. A sim argt apl^{s} to TaS. Pb IV (art. 50, p 17) will sgst the mode of f^{g} αa, βa, and γa,—also L_3a, L_1a, and L_2a.

28.—Is sim to pb^{s} 20-27, two pj^{ns} being g^{n} to f the third.

29.—The ptn of a' bet TaH and g is the h leg of a rt △ wh v leg is the ⊥ from TaV to g. αa is the acute ∠ in this rt △ adj to the h leg.

30.—The ⊥ from TaH to g_2 is one leg of a rt △; γa is the acute ∠ opp this leg. The other leg shows the dst from (TaH)$'''$ to TaS.

31.—The ptn of a'' bet g and g_1 is one leg of a rt △ of wh L_2a = the hpt, and the adj acute ∠ = βa.

32.—The dst of TaV from g_1 = one leg of a rt △ and γa = the opp acute ∠. This det^{s} the l of the other leg and ∴ the psn of TaS.

33.—D any aux line ∥ to the rq^{d} line, and then thr TaS ps the rq^{d} line ∥ to the aux line.

34.—The ptn of a''' bet TaS and g_2 is one leg of a rt △ of wh L_2a is the hpt. The l of the other leg = the dst of TaH from g_2.

35.—The ⊥ from TaV to g is one leg of a rt △ of wh L_3a is the hpt and the other leg = the ptn of a' bet TaH and g.

36.—L_2a = the hpt and the ⊥ from TaS to g_2 is one leg of a rt △ wh other leg is the ptn of a' bet g_2 and TaH; nb that TaH is v^{y} pj^{d} wh a'' meets g.

37.—D any aux line ∥ to rq^{d} line and f αa. αa with L_2a will enable TaH to be lc^{d}. Pb 37 is thus rd^{d} to pb 29.

38.—Pb 38 is es^{y} sim to pb 37 in analysis.

39.—Two pt^{s} on the rq^{d} line are g^{n}.

40.—TaV with αa g^{s} L_3a. L_3a with βa g^{s} the l of the ptn of a'' bet TaV and g. a'' may then be d^{n}, then a'. See fig. 23.

41.—L_3a and *αa* g l of ptn of *a*′ bet T*a*H and g. *a*′ may then be d^n. *a*′ and L_3a being k^n, *a*″ may ry be d^n.

42.—L_2a is the hpt of a rt △ and *βa* the ∠ adj to the leg wh = the ptn of *a*″ bet g and g_1. *a*″ may then be d^n. T*a*H being g^n, *a*′ is ry d^n. See fig. 24.

43.—D *b*, an aux line, ∥ to *a* and put T*b*H at *q*′, so that *a*′ will cncd with *b*′. *b*′ and *b*″ may be d^n by pb 40 or pb 41. L_3a and *βa* dt the dst of T*a*H from g; T*a*H may then be ry lc^d on *b*′. Then d *a* ∥ to *b* thr T*a*H. See fig. 25.

44.—D *b* ∥ to *a* putting T*b*V at *q*″, thus getting *a*″. L_1a and *βa* dt the dst of T*a*S from g_1. Lc T*a*S and thr it d *a* ∥ to *b*. See fig. 26.

45.—L_3a and *βa* dt the dst of T*a*H from g. L_3a and *αa* dt the l of the ptn of *a*′ bet T*a*H and g while L_2a and *αa* dt the l of that ptn of *a*′ bet T*a*H and g_2.

46.—Dt *q*′ then r *a* on P*q*H ∥ to V and dt dst of T*a*V from g, then r *a* back to ol psn and d *a*″.

47.—Dt *q*′, then r *a* on P*q*S ∥ to H, then dt T*a*S and d *a* thr T*a*S and *q*.

48.—L_1a and the ptn of *a*′ bet g and g_2 dt *αa*. *a* may then be d^n by pb 46.

49.—T*A*S and T*A*V meet on g_1. Δ*A*H may be f^d by using an aux pl *B* ⊥ to T*A*H and f^g the ∠ bet T*BA* and T*B*H. For τ*A*HV, etc., see pb VI, art 52, p 18.

50.—Rvrs the pcd^e of pb 49.

51.—Rvrs the pcd^e of pb 49.

52.—Rvrs the pcd^e of pb 49.

53.—T*a*V is a pt of T*A*V.

54.—Rvrs the pcd^e of pb 53.

55.—First f T*A*V, then thr *q* d an aux line *b* lying in *A*. *q*′ is on *b*′. F *b*″, then *q*″.

56.—D *b* so that *αb* = *αa*, and *βb* = *βa*; then d *a* ∥ to *b* thr *q*.

57.—Thr *q* pass an aux pl *B* ⊥ to T*A*H. R the rt △ wh s^s are T*B* HV*A* into H on T*B*H as axis, n^g that *q* is on T*BA* and that P*q*H will r ∥ to T*A*H. The dst of *q* from T*A*H is thus g^n and the psn of q_rHT$_{AH}$ may ry be dt^d. See fig. 27.

58.—F T*A*V by ps^g thr q any aux line b lying in A. This will rd the pb to pb 57. q_rHT$_{AH}$ may be f^d without first f^g T*A*V, by dp^g from q a $\perp$ on T*A*H, and f^g its l by pb IV, art 50, p 19. See fig. 28.

59.—This pb is sim to pb^s 50 and 57.

60.—This pb is sim to pb^s 57 and 58.

61.—q'' is on a line thr q' $\perp$ to g, and on one thr q_r VT$_{AV}$ $\perp$ to T*A*V.

62.—F T*A*H by rvrs^g pb 49. Then f q'' and n that a line thr q $\parallel$ to T*A*H will, when r^d into V on T*A*V as axis make with T*A*V an $\angle$ = to τAHV.

63.—T*A*V is $\perp$ to the line thr q'' and q_rvT$_{AV}$. Sp t to dnt the pt wh T*A*V crs^s this line. Then the △ $q_{rV}tq$ is isos and has its vx at t. Its base is the hpt of the rt △$q_{rV}qq''$. R this △ into V on $q_{rV}q''$ as axis, f t, and then d T*A*v, after wh T*A*H may ry be d^n.

64.—The $\angle$ $q\ q_{rH}\ q'$ is ½ Δ *A*H.

65.—The dir^n of T*A*v is k^n (see pb 63). T B an aux pl $\parallel$ to A T*B*V is $\parallel$ to T*A*V and Δ*B*H = Δ*A*H. D T*B*V, then with that and Δ*B*H d T*B*H, then f Δ*B*V. K^g Δ*A*V, the pb becomes sim to pb 64.

66.—The dir^n of T*A*H is k^n (see pb 63). T B and Δ*B*H (wh = Δ*A*H) as in pb 65 and then slv as in pb 64.

67.—In the rt △qq_rq'', the $\angle$ $qq_r\,q''$ is ½ Δ*A*V and the leg qq'' = the $\perp$ from q' to g. The leg $q''q_r$ may then be dt^d, after wh q'' may be lc^d, and the pb rd^d to pb 63.

68.—F the r^d psn^s of any two pt^s on a. T*a*H and T*a*V are u^y as cnv as any.

69.—T*a*H is a pt of T*C*H, etc.

70.—Sim to pb 69.

71.—Thr q d an aux line $\parallel$ to a, or else one meeting a. This will rd pb 71 to pb 70 or pb 69. If the aux line meets a it is u^y most cnv to t it $\parallel$ to H or to V.

72.—Use an aux line crs^g a and b, or crs^g one and $\parallel$ to the other.

73.—Use any two co-pl^r lines carrying the three pt^s.

74.—Thr one of the three pt^{s} d an aux line $\parallel$ to H or to V and lying in the pl of the fig. Use this line as axis and r the fig $\parallel$ to H or to V.

75.—Thr any pt of *b* d an aux line *c* wh shall be $\perp$ to *A*. The cmpt of $\angle$ *bc* is the $\angle$ *bA*.

76.—See pb X, art 56, p 20.

77.—See pb XI, art 57, p 20.

78.—See pb XV, art 61, p 21.

79.—See pb XIV, art 60, p 21.

80.—Rvrs pb 79.

81.—F the dst of *q* from *B*. K^{g} this dst and the $\angle$ *aB*, the dst of *q* from T_{aB} (wh, nb, is a pt of T_{AB}) may ry be dt^{d}, then T_{aB}, then *a* may be d^{n}.

82.—D any aux line *b* lying in *A* and making with *B* the ds^{d} $\angle$ *aB*. Then thr *q* d *a* $\parallel$ to *b*.

83.—Thr *q* d c_1 $\parallel$ to *b*, and c_2 $\perp$ to *A*. *C* is the pl of c_1 and c_2.

84.—From any pt of *a* (except T_{aB}) dp a $\perp$ to *B* and dt its l and its foot in *B*. Its l will with Δ_{CB} dt the dst from its foot to T_{CB}. T_{CB} also goes thr T_{aB}. T_{CB} may then be d^{n} and it and *a* will be two lines of *C*.

On the Sln of Triedrals. In slv^{g} triedrals we u^{y} cncv the triedral so lc^{d} that one of the faces shall lie either in H or in V, and one of the edges be $\perp$ to g, thereby much facilitating the sln. Thus sp α to lie in H, and *b* to be $\perp$ to g, *w* being in the front part of H. Then the face γ is $\perp$ to V and $T_{\gamma V}$ makes with g the pl $\angle$ of the Δ B, while $\Delta_{\beta H} = \Gamma$.

A is then the only Δ to be f^{d} as the Δ bet tw pl^{s} both oblique to g. See fig. 29 for a triedral thus g^{n} and slv^{d}. The student may find a cdr^{n} of the crsp^{g} cases in sph^{c} Δ^{s} useful in aiding his imagination.

85.—T *w* anywh in front part of H. D *b* $\perp$ to G, lay off γ on (say) the left-hand s of *b*, *a* on the opp, and β beyond *a*. A brief cdr^{n} of the mode of slv^{g} a triedral as sh^{n} in fig. 29 will sgst the rm^{g} pcd^{e}.

86.—Lay off β in H with *c* $\perp$ to g and a_{dHc}. Since Γ is g^{n}, vT*a* may ry be d^{n} and vT*b* be f^{d}. The rest of the pb is ry slv^{d}.

87.—Lay off β as in pb 86. *a* is hTγ, and having *a* and $\Delta\gamma$H (= A), vTγ may ry be d^{n}. T*b*V is a pt of vTγ and may ry be f^{d} by the mthd sh^{n} in fig. 29. Dt in what cases there are two sln^{s}.

88.—Lay off *a* in H with *b* $\perp$ to g; hTβ and hTγ being thus g^{n}, together with the values of B and Γ, the triedral may ry be d^{n}.

89.—Lay off $\beta+\gamma$ in H with *b* $\perp$ to g; cstr B and put *a* in psn, thus f^{g} vT*c*. F also the dp^{d} psn of vT*c* when β is dp^{d} into H. This pt and the true psn of vT*c* are eqdist from T*a*g.

90.—Es^{y} sim to pb 89.

91.—Sp the edge *a* to be t^{n} $\perp$ to g, and β to lie in H. vTγ may ry be d^{n}. If a sph be d^{n} about any pt of *a* as ctr and two cones of r^{n} be cstr^{d}, one with its base in γ and its base $\angle$ = to B and the other with its base in β and its base $\angle$ = to Γ, a pl tan to these two cones will cntn *a* or be $\parallel$ to *a*. If the ctr of this sph be t^{n} at vT*a*, the v tr^{s} of these aux cones may ry be d^{n}, also the h tr of the cone wh base is in β. The v tr of the tan pl ps^{s} thr the apices of these cones, the h tr is tan to the base lying in β.

92.—Lay off *a* in H with the edge *b* $\perp$ to g, and cstr T*a*V so that when the triedral is completed γ shall lie in H. F vT*c*, and k^{g} thus one pt in vTβ and one in hTβ, also $\Delta\beta$H (= A,) TβV and TβH may ry be d^{n}.

93.—Cncv the triedral in psn with γ in H and *a* $\perp$ to g. R *a* into the pl of β on *c* as axis so as to get $a+\beta$, then r $a+\beta$ into H on *a* as axis, r^{g} so that $a+\beta$ shall cover γ. The l of *c* will then be sh^{n} and the dst from vT*c* to *b* may ry be dt^{d}. Sp a $\perp$ dp^{d} from vT*c* upon *b* when the triedral is in psn, and r^{d} into V about hP (vT*c*) as axis, and so as to lie on the opp s of this pj^{r} from vTβ. It and vTβ will then form with g a $\triangle$ wh int base $\angle^{s}$ on g will be rsp^{y} B and A, and whose ext $\angle$ at vT*c* will $\therefore$ = A+B. By cstr^{g} this $\triangle$, A and B may ry be f^{d} and the cstr^{n} pcd^{d} with by the u mthd. See fig. 30.

94.—The first ptn of the pcd^{g} may be as in pb 93. Then nb that if of a $\triangle$ EFG two s^{s} *e* and *f* be g^{n} and the $\angle$ (E−F), the $\angle^{s}$ E and F may be f^{d} by cstr^{g} a $\triangle$ in wh two s^{s} shall be rsp^{y} = to e and f and the incd^{d} $\angle$ = to (E−F). The ext $\angle$ adj to f will then = E, and the int $\angle$ opp f will = F. A and B being f^{d} by this mthd, the cstrn for pb 94 may pcd by the u mthd.

95.—If thr the edge c a pl be ps^{d} making with β the ΔA, this pl, the face β, and the pl in wh γ lies will form an isos triedral of wh β will be the base face. The bs^{r} of the Δ bet the aux pl and the pl in wh γ lies will be $\perp$ to β, and the Δ bet the aux pl and the face α = A−Γ. The pb may be most ry slv^{d} by t^{g} α in H with c $\perp$ to g.

96.—This is merely a pb in f^{g} the tr^{s} of g^{n} lines upon a g^{n} pl, and then f^{g} the true fig of the pgn of wh these tr^{s} are the vx^{s}.

97.—Sim to pb 96. See fig. 31.

98.—The dvp^{t} will cnst in f^{g} the true fig^{s} of the various lt faces of the trnc fig^{s} and then unfolding them into a flat surf, keeping their proper edges cntg. See fig 31 for the dvp^{t} of the lt surface of the trnc pyr.

99.—Sim to pb^{s} 96, 97, and 98.

100.—Lay out the lt surf of the pyr in its dvp^{d} form and thus find the sizes of the basal edges F wh the $\perp^{s}$ from the apex of the pyr upon the basal edges meet these edges, also the l^{s} of these $\perp^{s}$. K^{g} these and the psn^{s} their pj^{ns} upon the base will t, the alt of the pyr and the pj^{ns} of its apex are ry dt^{d}.

101.—The faces in order to make = Δ^{s} with the base must have = slh^{s}, and the pyr must $\therefore$ be circumscriptible about a cone of r^{n}. Any lt edge then must make = $\angle^{s}$ with the el^{s} of tan^{y} of the two faces in which this edge lies. A brief cdr^{n} will sh how to put these el^{s} in psn in the dvl^{d} lt surf, and then how to dt the l^{s} of the lt edges after the slh has been chosen.

102.—Sim to pb 101.

103.—Cdr the rl wh opp s^{s} of the ▱ must bear to the com line of the two faces in wh they lie.

104.—Cdr the rl^{s} sstn^{d} by the dg pl^{s} of the pyr to each other.

105.—The shadow will be dt^{d} by the tr^{s} of pl^{s} dt^{d} by the source of light and the two lt edges of the pyr which will cause the Δ bet these pl^{s} to be a max.

106.—Sim to pb 105.

107.—A cyl being merely the lim of a pm, the prin^{s} made use of in pb 96 will apply here. See fig. 32.

108.—Pb 108 bears the same rl to pb 97 as does pb 107 to pb 96. Nb that in dvp^{g}, if the sec pl had not been ⊥ to the el^{s} of the cyl, we should have had to use an aux pl ⊥ to the el^{s} and dvp the two trnc cyl^{s} thus obtained.

109.—Same cdr^{ns} as in pb 105. See fig. 33.

110.—Aux pl^{s} cncr^{g} in a line thr the apex of the cone ∥ to the el^{s} of the cyl will have st tr^{s} on both cv^{d} surf^{s}. The isn^{s} of these tr^{s} will be pt^{s} of the tr ds^{d}.

111.—From cdr^{ns} of el^{y} geom the tr is k^{n} to be a ⊙ wh ctr is the pj^{n} of the ctr of the sp upon the sec pl, and wh rad is ry dt^{d} when the l of this pj^{r} is cf^{d} with the rad of the sph. By r^{g} the sec pl into H upon its h tr as axis, this ⊙ may ry be d^{n} and then r^{d} back to its ol psn. Or aux pl^{s} cncr^{g} in any cnv diam of the sph, say the v diam, my be used and the isn^{s} of their tr^{s} on sph and sec pl will be pt^{s} of the trace ds^{d}. Or aux pl^{s} ∥ to H or to V may be used. This last mntn^{d} mthd is frequently very cnv. The shadow will be dt^{d} by the tr of the sph on the sec pl and that of the cone enveloping the sph and having its vx at the source of light.

112.—Use aux pl^{s} ⊥ to the axis of the cone. Had the cone been obl it would have been best to use pl^{s} cncr^{g} in the line ps^{g} thr the apex of the cone and the ctr of the sph.

113.—Use aux pl^{s} ∥ to the axes of the two cyl^{s}. See fig. 34.

114.—Use aux pl^{s} ⊥ to the axis of the hyperboloid. The hyperboloid is most cnv^{y} t^{n} so as to have its axis ⊥ to H or to V. See fig. 35 in wh the axis is t^{n} ⊥ to H.

115.—The ds^{d} pl *A* must be tan to two cones of r^{n} having their apices at *q*, the one having its axis ⊥ to H and its base ∠ = Δ*A*H, the other having its axis ⊥ to V and its base ∠ = Δ*A*V. ∵ the apices of these two cones cncd, in order to d this tan pl some line must be f^{d} tan to both cones. This may be done by cutting them both by some aux pl not ps^{g} thr the apex and then t^{g} a line tan to the tr^{s} of this aux pl on the two aux cones. Dt how many sln^{s}.

CHAPTER V.

PERSPECTIVE.

66.—By the **perspective** * of any figure is meant **the graphical representation of that figure upon any surface as it would appear when looked at through that surface treated ás transparent.** Theoretically it differs then only from the centric projection of the figure in that it is a combination of the two centric projections got from the two eyes of the observer as centers of projection instead of being a single centric projection. Practically, however, the distance between the two eyes of the observer is treated as negligible and a single centric projection is used.† A very little practice enables the observer to make this single projection serve equally as well as would a stereoscopic projection.

67.—Parallel projection, to which we have hitherto almost entirely confined our attention, being but a special and limited case arising under centric projection, it is necessary to consider somewhat in detail some of the complications of general centric projection which become eliminated in the change from that to the simpler case of parallel projection. A full consideration of the relations between figures and their centric projections constitutes what is known as **projective, or modern, geometry.** Into that, however, it is no part of our present plan to enter. We purpose merely to examine into a few of the most elementary facts such as we shall later find practical application for.

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* *Perspective* is derived from the Latin, *per*, through, + *speccio*, look.

† An instrument called the stereoscope is frequently to be found in collections of optical instruments, and is usually to be found discussed in text-books on optics. By its use the two slightly different central projections got from the two eyes of the observer as centers of projection are combined into one in such a way as to give a more complete appearance of solidity than a single perspective drawing can present.

68.—If, of two concurrent lines, both passing within finite distances of the observer and one of which is considered fixed, the point of concurrence be conceived to recede along the fixed line, the two lines will become less and less divergent until as the point of concurrence recedes to an immeasurably great distance the divergence of the two lines will become immeasurably small, and we shall be totally unable to distinguish the two lines from parallel lines, so far, at least, as any characteristics they present within a finite distance of the observer are concerned. It is because of this that we say that parallel lines are lines meeting at infinity. Moreover, the point of concurrence having receded to an infinite distance it may continue receding indefinitely and we shall in no wise be able to distinguish any change in the appearance of the two straight lines within finite space. Therefore instead of speaking of the *points* at infinity along any straight line which lies within a finite distance of the observer, we speak merely of the **point at infinity,** any one such point being, so far as any value it may have for us who remain in finite space is concerned, indistinguishable from any other.

69.—Likewise if the line of concurrence of any two planes be removed to an infinite distance from the observer, the two planes themselves continuing within finite distances of him, they will become indistinguishable from parallel planes. We may thus say that parallel planes are planes meeting at infinity, and just as in straight lines with regard to their point of concurrence when it is at infinity, so here with regard to planes meeting in a line at infinity, we are totally unable to distinguish any effect produced by varying the position of the line of concurrence of the movable plane with the fixed plane so long as that line remains at an infinite distance. We therefore look upon all points at infinity in a plane as being ranged in one straight line which we call the **line at infinity** for that plane.

70.—Now it is at once apparent that the centric projection of any point is the trace of its projecting ray upon the base of projection, and that the centric projection of the extremity of any line is the extremity of its centric projection. If, therefore, we take the projecting ray of the point at infinity of any system of parallel lines, the trace of this ray upon the base of projection is the projection of a point common to all of the lines and one to which all their projections converge. This projecting ray is, moreover, parallel to the lines themselves, since it meets them at their point at infinity. Since it passes through the center of projection we shall call it the **centric line** of the system. The centric projection of a system of parallel lines then is a system of concurrent lines whose center of concurrence is the trace of the centric line of the system upon the base of projection. The projections all terminating or vanishing at this point, it is called the **vanishing point** of the system. If the lines be parallel to the base of projection, their centric line will meet the base at infinity, and their projections will thus be parallel to the lines themselves, the vanishing point being the point at infinity for the system.

71.—Similarly the vanishing points of all lines lying in any system of parallel planes will all be ranged in a straight line (the base of projection being here presumed to be a plane), for the centric lines of all the various systems of parallels will all lie in a plane through the center of projection parallel to the various parallel planes, and the traces of these centric lines on the base of projection will all lie in the trace of this plane on that base. This plane we shall call the **centric plane** of the system, and its trace on the base of projection is called the **vanishing line** of the planes of the system, since upon this vanishing line lie the vanishing points of all lines lying in such planes.

72.—In the execution of a drawing in centric projection, or perspective as we shall call it throughout the remainder of this chapter, the base of projection is commonly taken to be a vertical plane, and is called the **picture plane** (denoted hereinafter by PP). The center of projection is usually called the **point of sight,** sometimes the **station-point,** sometimes merely the **station** (denoted hereinafter by s); its orthogonic projection upon PP is called the **center of the picture** (denoted as usual by s''); and its orthogonic projection upon the ground-plane (see below) is called the **foot of the station** (denoted hereinafter by s_1).

73.—Any plane perpendicular to PP is called a **normal plane,** and will be denoted hereinafter by N with a subscript prefixed to denote to what system the plane belongs and one suffixed to denote through what point it passes; thus, ${}_1N_q$ will denote a normal plane belonging to the first system and passing through q. What system shall be called the first system, what the second, etc., is of course a matter of choice. The centric normal plane of any system will have no subscript suffixed. Any line perpendicular to PP is called a **normal line,** and will be denoted hereinafter by n with a supscript suffixed to indicate through what point it passes; thus, n_q will denote the normal line through q. The centric normal line is called the **axis,** and will be denoted by n without any suffix. That portion of the axis extending from s to s'' is called the distance line because it gives the distance of s from PP. Its length is frequently called the **distance.**

74.—Among the various systems of normal planes, two have preeminent importance. These are the horizontal normal planes, denoted hereinafter by $_hN$, and the vertical normal planes, denoted hereinafter by $_vN$. The centric horizontal normal plane is called the **horizon plane** (denoted hereinafter by H) and its trace on PP is called the **horizon** of the picture (denoted hereinafter by h); that one supposed to be at the general level of the ground is called the **ground plane** (denoted hereinafter by G), and its trace on PP is called the **ground-line** (denoted as before by g). The centric vertical normal plane will be called the **meridian plane** (denoted hereinafter by M), and its trace on PP the meridian line (denoted hereinafter by m).

75.—Planes parallel to PP will be called **secondary planes**, and will be denoted hereinafter by V followed by distinguishing subscripts; thus, V_q will denote a secondary plane through *q*. The centric secondary plane will be denoted by V without any suffix. Any line or figure parallel to PP will be called a **secondary line or figure** and will be denoted by a subscript $_2$ prefixed to the symbol for the line or figure. The ratio which the perspective of any secondary sect* bears to the sect itself is called the **scale of the secondary plane** in which that sect lies.

76.—Horizontal lines inclined at 45° to PP we shall call **principal diagonals.** Their vanishing points are evidently on the horizon line and at distances from s″ equal to s″s; for this reason these vanishing points are known as **distance points:** the one to the left of s″ will be denoted by d_1, the other by d_2. Lines inclined at 45° to PP and lying in or parallel to M have their vanishing points on m, equally distant with d_1 and d_2 from s″; these lines we shall call **subordinate diagonals,** their vanishing points being called **lower and upper distance points,** and denoted herein by d_3 and d_4 respectively. The vanishing point of normals is evidently s″. The perspective of any figure we shall denote by the symbol for the figure followed by the subscript p,—thus, a_p denotes the perspective of the line a. A subscript $_v$ attached to the symbol for a line or for a plane will be used to denote the vanishing point or line (as the case may be) of the figure to whose symbol it is attached.

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*A sect is any definite portion of a straight line.

77.—Perspective drawing as practiced by the draughtsman is based on a few simple propositions some of which have already been stated. For convenience of reference we shall here state them in compact form. Their demonstration is left to the student.

78.—Proposition I. Every point in PP is its own perspective.

79.—Proposition II. If a is a straight line a_p is either a straight line or merely a point.

80.—Cor. The perspective of a sect is determined by the perspectives of its extremities.

81.—Proposition III. If q is on a, q_p is on a_p.

82.—Cor. I. If a and b concur in q, a_p and b_p concur in q_p.

83.—Cor. II. Perspectives of concurrent lines whose center of concurrence is in V are parallel.

84.—Cor. III. Perspectives of secondary lines are parallel to the lines.

85.—Cor. IV. Perspectives of normals vanish at s″.

86.—Cor. V. Perspectives of lines equally inclined to PP vanish at points equidistant from s″.

87.—Proposition IV. If any figure is symmetric about the axis, its perspective is symmetric about s″.

88.—Proposition V. If a lies in or is parallel to A, a_v lies in A_v, and if A and B are parallel A_v and B_v coincide, and conversely.

89.—Proposition VI. A_t* is parallel to A_v.

90.—Proposition VII. The perspective of any secondary figure is similar to the figure, and the ratio of similitude equals the scale of the secondary plane.

91.—Proposition VIII. If a straight line a is tangent to a curve k at the point q, a_p is tangent to k_p at q_p.

92.—We now are prepared to consider the problem of finding the perspective of any given figure. Since every figure is composed of points, our problem when reduced to its elements consists merely in finding the perspective of a single point; this as has been so often said consists merely in finding the trace on PP of the ray determined by s and the point. It is then merely the problem of finding the trace of a given line on a given surface. Suppose (fig. 8 bis) s and q both given, and s_1 and q_1 the traces made on any plane F by two lines through s and q parallel to each other and to PP. Also let F_t be here denoted by a. Conceive a plane passed through the lines ss_1 and qq_1. Its trace on F will be the straight line s_1q_1 which crosses a at w, and its trace on PP will be a line through w parallel to ss_1; where this trace is crossed by sq is the perspective of q. These considerations are entirely independent of the position of F, likewise of the direction of the lines ss_1. qq_1, and q_pw. It is readily apparent also that q_p will remain unchanged in position if F revolves on a, carrying with it the points s_1 and q_1, the lines ss_1 and qq_1 and the points s and q, provided these lines remain parallel to their original positions and retain their lengths unchanged. We may then evidently replace F by the revolved position of H as we use it in our ordinary work in Descriptive Geometry.

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* A_t will be used throughout the remainder of this chapter to denote the trace of A on PP.

93.—It is also evident that if through s_1q_1 parallel lines be drawn meeting a in r and t respectively, and on these lines s_3 and q_3 be taken so that $\frac{rs_3}{rs_1}=\frac{tq_3}{tq_1}$, then s_3q_3 will cross a at w; and that if s_3s_2 be taken parallel and equal to s_1s and on the same side of s_1s_3, and q_3q_2 be taken parallel and equal to q_1q and on the same side of q_1q_3, then will s_3q_3 pierce PP at q_p. This last statement is equally true whether or not s_3 is on the same side of a with s_1, provided that, when s_3 is on the opposite side of a from s_1, q_3 shall be on the opposite side of a from q_1.

94.—We are thus led to the following practical method of constructing the perspective of a point q when its projection on H (q') and its altitude from H (the length of $q'q$) are known. H being supposed revolved into PP about h, from q' draw a line perpendicular to h and equal to $q'q$, laying it off above q' if q is above H, below q' if q is below H, and mark the other extremity of the line so drawn q. Then draw from s (*i. e.*, from its revolved position as a point of H) a line to q' and where sq' crosses h erect a perpendicular to h. Where such perpendicular meets sq is q_p (see fig. 41). It will be noticed that when sq is nearly perpendicular to h this method gives extremely unsatisfactory determinations of q_p. In such cases it is better to lay off $q'q$ parallel to h as is done in the right-hand portion of fig. 41; that portion of h comprehended between the lines sq' and sq_r evidently equals the distance of q_p from h. The lines shown as mere construction lines in the figure should of course in any practical perspective drawing be drawn merely in pencil and afterward erased.

95.—Customarily the auxiliary projections, etc., are drawn on other sheets of paper than the one on which the finished perspective is to appear, and these are tacked to the draughting board in convenient positions (see figs. 41 and 42). Small scale plans and larger scale working drawings may be utilized in this way, and the results increased or diminished in the proper ratio to secure the perspective to the desired scale. Fig. 42 shows the mode of utilizing elevations when they are drawn to the same scale as the plans used as auxiliaries. In that figure s_1t evidently bears the same ratio to s_1q' as does sq_p' to sq', so that if in the lower right-hand portion of the figure, showing the side elevation, a vertical line be erected at t_1 as far from q as t is from q' in the plan, and s_1''' as far to the left of that line as s_1 is from t, and at the same height above the ground line that s is above the ground plane, lines from s_1''' to various points of the elevation will cut the vertical line aforesaid at the heights of the perspectives of those various points above the ground line; thus, for instance t_1q_1 gives the height of q_p above the ground line. Of course if the plan and the elevation are not drawn to the same scale the proper modifications in these distances to reduce all parts of the drawing to corresponding scales will have to be made. A careful consideration of similar figures will be all that is necessary in order to do this correctly.

96.—The method above indicated serves very well when the perspective sought is that of a figure having few of its outlines and important points lying in the same plane or in systems of parallel planes. When, however, its bounding surfaces lie largely in systems of parallel planes as is the case in the greater number of practical applications of the principles of perspective in architectural drawing, we find it in most cases useful to find the vanishing-lines and points of the various systems of parallel planes and parallel lines, and to locate the perspectives of points largely by the use of auxiliary lines through the points. This mode is particularly useful when the points whose perspectives are sought occur in ranges, or rows. In the application of the method here indicated, we need to know how **to lay off on a line whose perspective is given the perspective of a given length,** also how **to divide a line whose perspective is given into parts proportional to given sects, and to determine the perspectives of the points of division.**

97.—If, in fig. 37, q denotes any point on a and the trace and vanishing point of a are as indicated, we may proceed thus to find the distance of q from a_t:—From s draw a line to a_v; it will be parallel to a. The ray from s to q pierces PP at q_p. The triangles sa_vq_p and qa_tq_p are similar. Therefore the distance of q from a_t is to the distance of s from a_v as the distance of q_p from a_t is to the distance of q_p from a_v. But if through a_t and a_v two parallel lines be drawn in PP and any line be drawn in

PP through q_p cutting the line through a_t at r and that through a_v at w, a_tr and a_vw will be in the same ratio, and will therefore be equi-multiples of the distances of q and s from a_t and a_v respectively. Any such point as w if taken so that a_vw equals a_vs is called a **measuring point** for the system of parallel lines vanishing at a_v. If taken so that the distance a_vw equals n times the distance a_vs, we shall call it an n-measuring point for the system. The line drawn through a_t parallel to a_vw is called a **measuring line** for the line a. In case the line a is a secondary line, the perspective of any section of it bears the same ratio to that section as the distance of PP from V bears to the distance of a from V.

98.—Consider next **the problem of finding the vanishing point and the trace of any line whose perspective is given,** when the distances of two of its given points from a given centric normal plane are given. Let a denote the given line (see fig. 38), q and r be the given points, ${}_1$N the given centric normal plane, and d_q and d_r the distances of q and r respectively from ${}_1$N. Suppose ${}_1N_q$ and ${}_1N_r$ drawn; their traces will be parallel to that of ${}_1$N and at distances from it of d_q and d_r respectively. Suppose any plane A passed through a. A will cut out of ${}_1N_q$ and ${}_1N_r$ lines passing through q and r respectively and parallel to each other and to ${}_1$N. Their traces then will lie in the traces of ${}_1N_q$ and ${}_1N_r$ respectively, also in A_t, while their vanishing point is in the vanishing line of ${}_1$N, which it will be remembered is the same as the trace of $_1$N. Suppose *w* the vanishing point of these two lines. Their perspectives pass through q_p and r_p respectively, and their traces are where the lines wq_p and wr_p cross the traces of $_1N_q$ and $_1N_r$ respectively. Through these two points A_t passes and where A_t crosses a_p is a_t. A_v passes through *w* and is parallel to A_t, and where A_v crosses a_p is a_v. Notice that since *A* is indefinite in position, *w* may be taken anywhere on the vanishing line of $_1$N.

99.—The solution of the problem **to divide a sect whose perspective is given into parts proportional to given sects, and to find the perspectives of the points of division** depends upon the fact that parallel lines intercept proportional parts on traversers (or transversals as they are frequently called). Suppose a_p (fig. 39) to be the perspective of the given line, q_p and r_p the perspectives of the extremities of the given sect, and a_v as indicated. Any line through a_v will be the vanishing line of some plane *A* containing the given line *a*, and any line through any point of a_p parallel to A_v will be the perspective of some line in the plane *A* parallel to A_v and therefore parallel to PP; its perspective therefore will be similar to the line itself. Then through a_v we may draw any line whatever and take it for the vanishing line of some plane *A* containing the line *a*, and through either extremity of the sect to be divided, *q* say, draw a line *qx* parallel to A_v (and therefore parallel to PP); on the line so drawn if parts be laid off proportional to the parts into which the given sect is to be divided and in the same order, lines through the points of division and the points sought on the given sect will be parallel in space, and in the perspective will converge to some point on A_v.

100.—It frequently happens that lines are so slightly inclined to PP that their vanishing points are off the sheet. In that case, in order to utilize the vanishing point, we find it very convenient to make use of the fact that the bisector of any inscribed angle bisects the intercepted arc. The manner of doing this will now be explained. In fig. 36, let v denote a vanishing point, and let o denote the center of any circle struck through v. Then if n^2+1 denote the radius of this circle, any two points on its circumference symmetric about the diameter through v and distant $2n$ from such diameter will lie on a chord distant n^2-1 from o, and $2n^2$ from v (provided, of course, that o is between v and such chord). Suppose q and r to be two such points, and that their chord crosses the diameter vo at w, also that t is the other end of such diameter. The angle qtw is then $\tan^{-1}n$, and the angle qtr is $2\tan^{-1}n$. Any ordinary T□ with fixed head may evidently be easily modified so as to make use of this principle. Pins should be driven into the draughting board so as to be symmetrically disposed about some line through the vanishing point and each distant from that line $2n$, and the line joining the pins distant $2n^2$ form the vanishing point. The angle qtr

should be cut into the T⊐ head as shown in the figure, and should have a chord of 4n at a distance of 2 from the vertex of the angle. The distance of the vanishing point from the chord joining the pins is evidently $\frac{2n^2}{4n}$, or $\frac{n}{2}$, times the distance between the pins, and n^2 times the distance *wt* in the figure. A slight examination of the various values which n may conveniently be given in practice may not be entirely worthless. Let it be assumed that the T⊐ head is 10″ by 2¼″, and that the blade is 3″ wide; also that it is not desirable to have the distance *wt* more than 1″, or the distance between the pins less than ½″ or more than 3¼″, and that there is a vacant portion along the edge of the draughting board at least 1½″ wide into which the pins may be driven without injury to the drawing. Then we shall have that *vw* can not be less than $\frac{1''}{2} \times \frac{n}{2} \left(= \frac{n''}{4}\right)$, or greater than $3\frac{1}{4}'' \times \frac{n}{2} \left(= \frac{13n''}{8}\right)$, or than $(n^2)''$ in case *wt* is not to exceed 1″. Then when
n = 5, *vw* minimum = 1¼″, and *vw* maximum = 8⅛″; when
n = 32, *vw* minimum = 8″, and *vw* maximum = 52″; and when
n = 200, *vw* minimum = 50″, and *vw* maximum = 325″.
It is readily apparent that a very small assortment of modified T⊐s will be sufficient for any practical work, since two lines each ten inches long and separated at their more remote ends by one inch would approach each other by only one-thirtieth of an inch if their point of convergence were distant three hundred inches, a rate of convergence too slight to be perceptible to any

except the most highly trained eyes. Instruments embodying the principle here discussed are to be had of dealers in draughting instruments. They are sold under the name of **centrolineads.**

101.—Lines converging to a point off the sheet may also be drawn by making use of the fact that concurrent traversers (or traversals) make proportional intercepts on parallels, so that if the intercepts made on any pair of parallel lines by any pair of lines meeting at the vanishing point be divided proportionally, lines through the corresponding points of division will concur at the vanishing point. Strips of paper graduated to different scales may be so tacked to the draughting board as to be parallel and to have equal numbers of graduations between any pair of given lines concurring at the vanishing point. By the exercise of a little judgment, lines may be drawn through given points so as to pass through corresponding points of the scales and thus be directed to the vanishing point. The points of division on the strips ought not to be more than one-eighth of an inch apart. Lines graduated to various scales and used as indicated in the discussion above given are called **vanishing scales.** See fig. 40.

162.—We come now to the final and practical problem in perspective drawing,—viz., how best to dispose PP, s, etc., with regard to the object whose perspective is to be drawn. This object we shall hereafter call simply *the object.* The disposition will be governed chiefly by two considerations,—first, the production of a good pictorial effect; second, ease of making the drawing.

103.—With regard to the first it is to be noted that persons examining a picture will in the majority of instances direct their attention chiefly to the middle portion of the sheet on which the picture is. In examining it to any considerable extent, they will usually change their distances from it as well as shift their positions parallel to it; *i. e.*, the positions of their eyes will only approximate to that of s. A brief consideration will suffice to show that the farther s is from PP in comparison with the distance of PP from the object, the less would the perspective be modified by a definite amount of shifting of s, and consequently the less distorted will the picture appear when viewed from any point within a given distance from s. We conclude then that s should be taken as far from PP, and s'' as nearly midway from side to side in the picture, as convenience will permit. A good practical rule is to take s at a distance at least seven-tenths of the width of the picture.

104.—In so disposing of PP with regard to the object as to facilitate the work of drawing, notice that if PP be parallel to any set of lines of the object their perspectives will be parallel to the lines themselves, and hence may very readily be drawn. The lines of architectural constructions usually have three principal directions,—one vertical, the other two horizontal and perpendicular to each other.

105.—PP may be placed parallel to two of these, one vertical and the other horizontal; the perspective is then called **parallel, or one-point, perspective,** only one of the three principal directions giving a finite vanishing point, which is evidently s″. This disposition evidently gives the easiest possible perspective constructions. One-point perspective produces the best pictorial effect when the parts of the object are symmetrically disposed about some open space down which the observer is conceived to be looking, as in the case of the buildings on the two sides of the street; it may also be used satisfactorily when the object is entirely or almost entirely below H, so that some of the lines in the perspective will be shown converging towards s″.

106.—If PP is placed parallel to only one of the principal directions, the perspective is called **angular, or two-point, perspective,** the other two directions both giving finite vanishing points. In this case the drawing will be facilitated by so placing PP that the two directions oblique to it shall be equally inclined to it. Where but one object is to be represented the best pictorial effect is produced by two-point perspective, except in the case before mentioned where the object is entirely or almost entirely below H.

107.—If PP is placed parallel to none of the three principal directions the perspective is called **oblique, or three-point,** this arrangement giving three finite vanishing points. If PP is equally inclined to these three directions, their vanishing points will evidently be equidistant from s″. The case of three-point perspective will not, however, need often to be considered in practical work.

108.—If shadows are to be shown, the work of obtaining them will be much facilitated, of course, by taking the rays of light parallel to PP if they are to be considered parallel as they may be when coming from a source at a very great distance. In case they come from a source so near as to make them essentially divergent, the placing of the source will have to be determined by the particular features of the problem.

109.—After the center of the picture, the horizon, and the ground line have been selected, the vanishing points of the principal lines of the object are usually next determined. The vanishing point of any system of parallel lines, it will be remembered, is the trace on PP of the centric line of the system, *i. e.*, of the line through s parallel to the lines of the system. The lines drawn from s to the vanishing points of any three systems form then the three edges of a trigonic pyramid whose base lies in PP, whose face angles are given (being the angles between the directions of the three systems), and the orthogonic projection of whose vertex s on its base PP is at s″. A brief consideration will show that two of the corners of its base may be chosen at random, and that the third is determined by these two and s″, also that the altitude ss″ is also determined. Then having chosen the vanishing points of two systems of principal lines determine that of the third, and find also the distance ss″. Next determine the distance points, the measuring points of the principal systems of lines, the vanishing lines of the various oblique planes, the vanishing points of oblique lines, etc. The determination of these will consist in the application of simple principles of Descriptive Geometry, such as have heretofore been amply discussed and illustrated and so need not here longer detain us, and of the fundamental principles of Perspective as heretofore laid down. These various points and lines having been determined, the next thing to be drawn is the perspective plan.

110.—The perspective of the ground plane occupies that portion of PP between g and h, and of course the farther away the lines in it are from PP, the more is their perspective narrrowed and confined vertically, and the more oblique to each other do the perspectives of intersecting lines become. For this reason it is often very convenient to make use of what are called **depressed perspective plans,** *i. e.*, plans drawn in perspective on horizontal planes depressed to a convenient extent below the ground plane of the picture. These may in most cases easily be drawn at a sufficiently great distance below the ground plane to put the depressed plan entirely below g in the picture, thus permitting it to be drawn on a sheet separate from that on which the picture is to be drawn. This permits the depressed plan to be saved, and used in drawing other perspectives if desired. Elevated perspective plans may of course equally well be used. The use of a depressed perspective plan is illustrated in fig. 43, which shows the construction of the perspective of a Greek cross on a square pedestal, the face of the cross being taken at an inclination of 30° to PP, and the front vertical edge of the pedestal being taken in PP. The vertical dimension of the pedestal is taken equal to the thickness of the cross, the horizontal dimension three times as great, and the bevel on the upper portion of the pedestal has a vertical extension equal to one-fourth of the altitude of the pedestal. The shadow is cast by parallel rays, parallel to PP, the sun being taken at an altitude of 60°.

111.—As this problem illustrates a considerable number of the principles already discussed it may not be amiss to examine it somewhat in detail. The horizon line having been chosen and s'' thereon, s is located at as great a distance as convenience permits. From s lines are then drawn meeting h at 45° in the points d_1 and d_2. The point d_3 falls on the sheet at s and d_4 falls diametrically opposite and equidistant with s from s''. The vanishing line for the front face of the cross is a vertical line, since the face of the cross is vertical. To find it a horizontal line is drawn through s meeting h at an angle of 30° at the point marked *q*. The vertical line q_1q_2 through this point is then the vanishing line of the system of planes parallel to the face of the cross, since this face makes with PP the angle 30°. Likewise is found the vanishing line of the side of the cross, it being the vertical line through the point marked r. The front bevel of the pedestal cuts out of the side face of the pedestal a line elevated 45° above a horizontal; to find the vanishing point of such line, s is revolved into h about r as center, and through the point u thus obtained a line is drawn at an angle of 45° to h to meet the vanishing line of the side face of the pedestal at the point r_2. Any horizontal line in the front bevel vanishes at q; the vanishing line of this bevel is therefore the straight line through r_2 and q. Likewise the vanishing line of the left-hand bevel is the line through r and q_1, that of the right-hand bevel is rq_2, and that of the rear bevel is $r_1 q$. Since the intersection of the front and the right-hand bevels lies in both of those planes, its vanishing point will be at z where their vanishing lines cross. Likewise the vanishing point of the intersection of the rear and the left-hand bevels is at x where the vanishing lines of those two surfaces cross; xz is therefore the vanishing line of that diagonal plane of the pedestal which contains the front vertical edge. This vanishing line might equally well have been found by drawing the bisector of the angle qsr, and through the point y where it meets h erecting a line perpendicular to h.

At w_1 is a half-measuring point for lines vanishing at q, the distance qw_1 being one-half of qs; and in order to minimize the work of drawing the depressed plan, since in its principal outlines the plan of the base is square, a half-measuring point (call it w_4) for lines vanishing at *r* was taken at a distance to the right of *r* equal to one-half the length *rs*. As this falls outside the plate-border it is not shown in the figure, but it is the point to which the lines 4-8, 3-7, and 2-6 converge.

The ground line 22-15 was taken at the desired distance below h, and the auxiliary ground line 1-5 was chosen at a sufficient distance below the ground line to permit the depressed plan to appear entirely below the ground line. The point 15 having been selected for the front corner of the base (it, of course, is entirely independent of the line sd_2 in the figure), from the corresponding point 1 on the auxiliary ground line lines are drawn toward *q* and *r* to serve as the front left-hand and front right-hand outlines of the depressed perspective plan of the base. Along the auxiliary ground line the distance 1-5 is then laid off, that distance being one-half the horizontal dimension of the pedestal. A line from 5 to w_1, the half-measuring point for lines vanishing at *q*, determines on the line 1*q* the depressed plan of the left-hand corner of the base (at the point 12) while one from 5 to w_4 determines on the line 1*r* the depressed plan of the right-hand corner. Lines from these two points toward *r* and *q* respectively complete the remainder of the outline of the depressed plan of the pedestal. Likewise the remainder of the depressed plan of the figure is drawn, the pillar of the cross having for its horizontal dimension one-third of that of the pedestal, and the bevel one-fourth of that of the pillar. 11-13 is the depressed plan of the upper edge of the front bevel, and 9-10 is that of the right-hand side of the pillar.

In constructing the perspective of the cross itself, use is made of the perspectives of the traces made upon the plane of the right-hand side face of the pedestal by the various outlines vanishing at q. Thus the perspective of the trace made upon this plane by the front edge of the top of the pillar is at 21. This point is found by laying off on the vertical from 15 the distance of that horizontal line above the ground plane, thus obtaining the point 20, then drawing the line 20-r, and taking the point 21 where the line 20-r is crossed by the vertical line erected at 7 in the depressed plan. Where the line 21-q is met by the vertical from the point 9 in the depressed plan is the perspective of the front upper corner of the pillar. That this point falls on the line ur_2 is, of course, merely a matter of luck, as also it is that the point 20 comes near the line qr_2.

In getting the perspective of the outlines of the shadow, use is made of the fact that since the rays are parallel to PP the shadows of vertical lines on horizontal planes are parallel to g and that their perspectives and those of the rays are thus parallel to the shadows and rays themselves. Thus, the perspective of the shadow of 16 falls at 22, and that of the shadow of 23 falls at 24.

Exercises. The short collection of exercises herewith given will serve as samples and may be readily added to at will by the instructor. As they embody only applications of the common principles and operations of Descriptive Geometry, together with those of Perspective already amply illustrated, explanatory notes are deemed unnecessary. In the exercises below presented, when a line is said to be given it will be understood to be given by its trace and its vanishing point; correspondingly a plane is to be understood to be given by its trace and its vanishing line; and a point may be taken to be given by the intersection of a line and a plane, or by that of two lines, or by the ray through s and the distance of the point from s. The student will find it a useful exercise to solve the proposed problems with the given points taken as given in these several ways. When anything is said to be required, if position is concerned then its perspective is to be understood to be required; if the size is called for, the true size is wanted.

116.—Given A_v, A_t, B_v, and B_t; find TAB.

117.—Given A_v, A_t, b_v and b_t; find TbA.

118.—Given a_v, a_t, b_v and b_t; determine whether or not a and b intersect, and if so at what point.

119.—Given two windschief lines, to pass a plane through one parallel to the other.

In the remaining problems, s' and the distance s"s are also to be understood to be given.

120.—Find the angle between two given lines.

121.—Find the angle between a given line and PP.

122.—Draw through s a plane perpendicular to a given line.

123.—Draw through s a line perpendicular to a given plane.

124.—Find the diedral between two given planes.

125. Find the distance from s to a given plane—to a given line.

126.—Draw through a given point a line perpendicular to a given plane.

127. Draw through a given point a plane perpendicular to a

given line.

128.—Find the distance from a given point to a given plane.

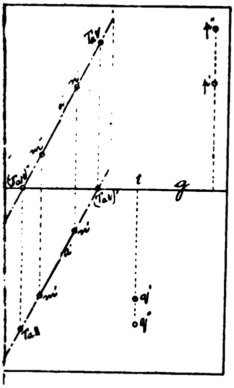
129.—Find the distance from a given point to a given line.

130.—Find the projection of a given sect upon a given plane.

131.—Find the angle a given line makes with a given plane.

132.—Draw through a given point a line parallel to a given plane to meet a given line.

133.—Find the distance between two windschief lines and draw the shortest line joining them.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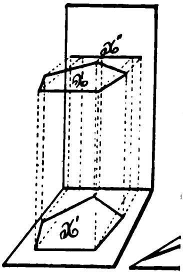
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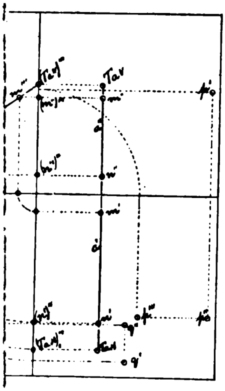
Fig. 5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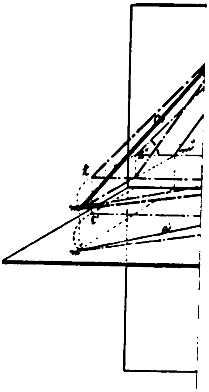
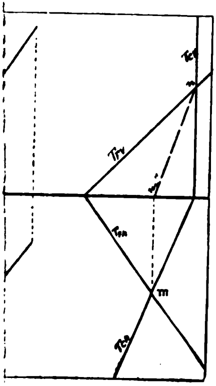
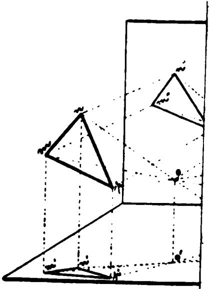
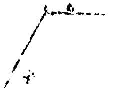
Tah
p'''
p'
q'


t
t'


Tcv
n
Trv
n'
Trh
m
Tch





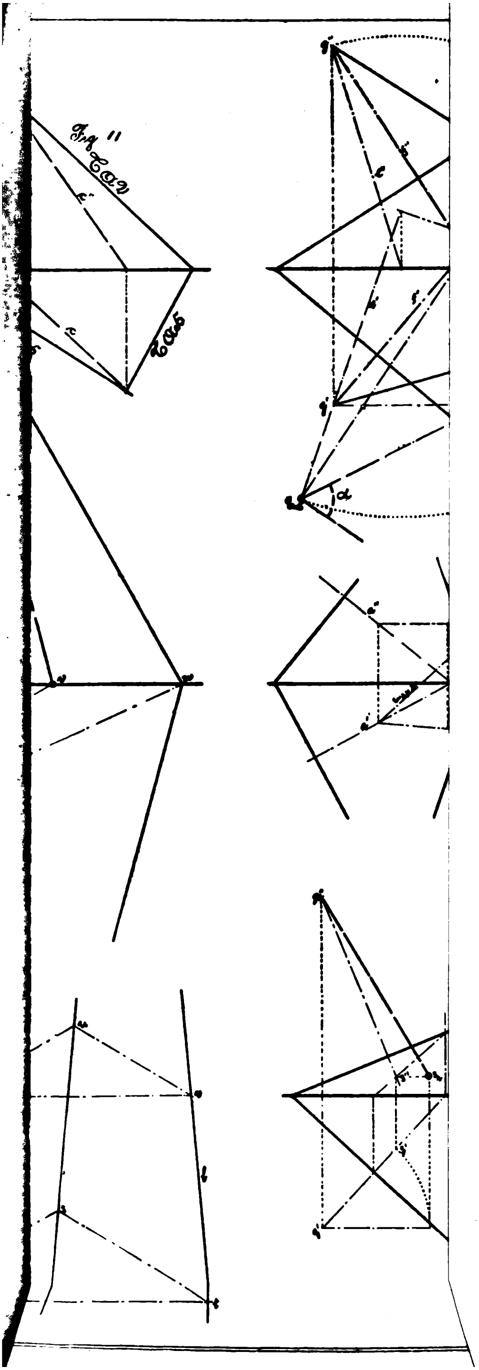
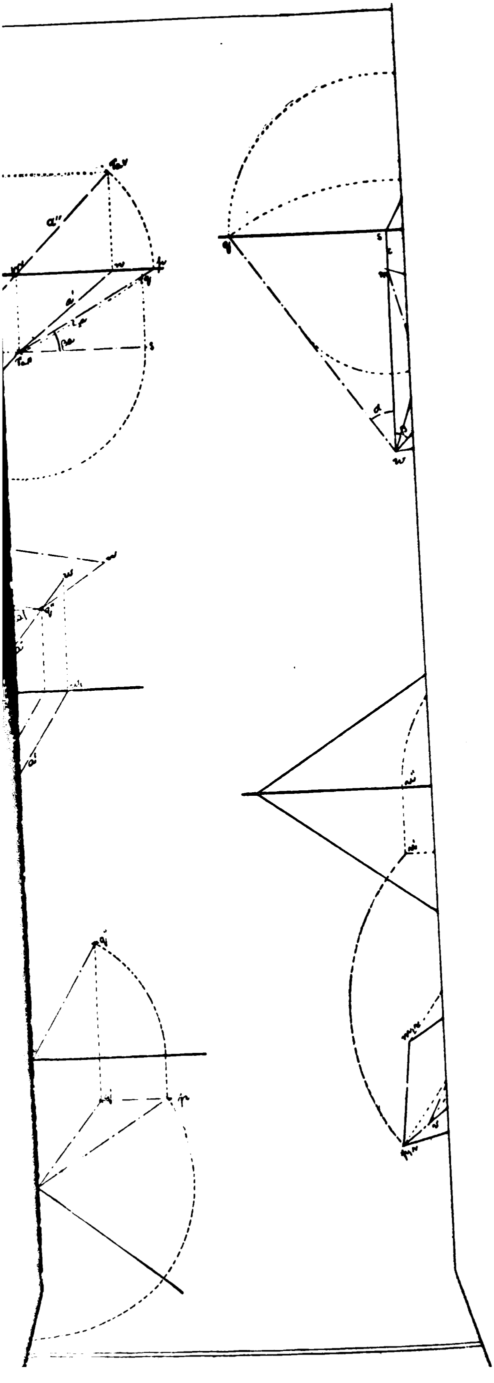
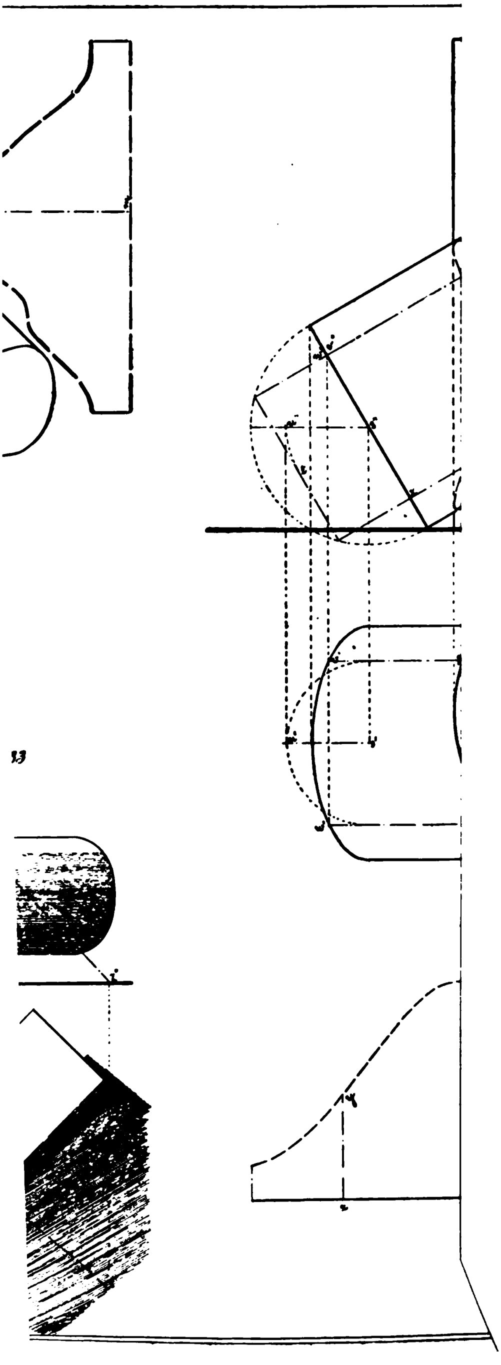


Fig 11
Tαa
Tαb
c''
c
α


a''
a'
w
s
q
d
w
a'
q''
p



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